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Feedback stabilisation of a sterile insect control system: Applications to mosquito-borne disease control

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Mots clés : Maladie vectorielle, Contrôle de la population de moustiques, système dynamique de contrôle, Conception de feedback, Methode backstepping, conception d'observateur, Stabilité au sens de Lyapunov, Equations de réaction-diffusion

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FEEDBACK STABILISATION OF A STERILE INSECT CONTROL SYSTEM: APPLICATIONS TO MOSQUITO-BORNE DISEASE CONTROL

Abstract

The sterile insect technique (SIT) has recently become one of the methods used for controlling mosquito populations. This technique involves the release of sterile mosquitoes that compete with wild males for mating with females. The mathematical study of this control system aims to develop strategies that optimize the release of sterile insects, stabilize the population to zero, and prevent resurgence. This is the objective of the present work. The challenge of stabilizing the system to zero lies in the fact that, as the population approaches zero, fewer releases are needed to maintain the population close to zero asymptotically. However, sufficient sterile insects must still be released to counteract, in particular, disturbances in the dynamic parameters. In control theory, many tools exist to address this problem, but they often fail to ensure the positivity of the control function. In this work, we apply the backstepping method to design a nonnegative feedback control law that addresses the problem of robustness and global asymptotic stability of the dynamics to zero. Additionally, we study other linear feedback control laws that can globally stabilize the control system. However, implementing these feedback control laws requires measuring the system states. We address this issue by designing an observer that, based on measurements of sterile and wild male mosquitoes, enables the estimation of the other components of the state of the system.

The implementation of SIT for controlling mosquito populations in practical situations also requires considering the spatial dispersion of adult mosquitoes. Recent advance in using drones to release sterile male mosquitoes have improved the precision of the release, which helps improve their dispersion.

In our study, we consider a system of reaction-diffusion equations that models the spatial dispersion of mosquito populations in a two-dimensional bounded and smooth domain. We study sterile male releases within this framework. Under Neumann boundary conditions and assuming the same diffusion coefficient for sterile and wild males, we design nonnegative feedback laws which globally asymptotically stabilize the population to zero over the entire domain.

Keywords: Vector borne disease, Mosquito population control, Dynamical control system, Feedback design, Backstepping feedback, Observer design, Lyapunov stability, Reaction-diffusion equations

Résumé

La technique des insectes stériles (TIS) est récemment devenue l'une des méthodes utilisées pour le contrôle des populations de moustiques. Cette technique consiste à relâcher des moustiques mâles stériles qui entrent en compétition avec les mâles sauvages pour s'accoupler avec les femelles.

L'étude mathématique de ce système de contrôle vise à développer des stratégies optimisant le lâcher d'insectes stériles, stabilisant la population près de zéro et empêchant toute résurgence. Le défi de stabiliser le système autour de zéro réside dans le fait que, à mesure que la population approche de zéro, moins de lâchers sont nécessaires pour la maintenir asymptotiquement proche de zéro. Cependant, il est crucial de relâcher un nombre suffisant d'insectes stériles pour contrer, en particulier, les perturbations des paramètres dynamiques. En théorie du contrôle, de nombreux outils existent pour résoudre ce problème, mais ils ne garantissent pas souvent la positivité de la fonction de contrôle.

Dans ce travail, nous appliquons la méthode de backstepping pour concevoir une loi de contrôle feedback positive, répondant au problème de robustesse et de stabilité asymptotique globale des dynamiques vers zéro. De plus, nous étudions d'autres lois de contrôle feedback linéaires capables de stabiliser globalement et asymptotiquement le système de contrôle vers zéro. Cependant, la mise en œuvre de ces lois de contrôle nécessite de mesurer les états du système. Nous abordons ce problème en concevant un observateur qui, à partir des mesures des moustiques mâles stériles et des mâles sauvages, permet d'estimer les autres composantes de l'état du système.

La mise en œuvre de la TIS pour contrôler les populations de moustiques dans des situations pratiques nécessite également de prendre en compte la dispersion spatiale des moustiques adultes. Les avancées récentes dans l'utilisation de drones pour relâcher les mâles stériles ont amélioré leur dispersion.

Dans notre étude, nous considérons des systèmes d'équations de réaction-diffusion modélisant la dispersion spatiale des populations de moustiques dans un domaine borné et lisse à deux dimensions. Nous étudions les lâchers de moustiques stériles dans ce cadre. Sous des conditions aux limites de Neumann et en supposant le même coefficient de diffusion pour les mâles stériles et sauvages, nous avons construit des lois de contrôle feedback stabilisant globalement et asymptotiquement la population vers zéro sur l'ensemble du domaine.

Mots clés : Maladie vectorielle, Contrôle de la population de moustiques, système dynamique de contrôle, Conception de feedback, Methode backstepping, conception d'observateur, Stabilité au sens de Lyapunov, Equations de réaction-diffusion

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Chapter 1

Introduction and outlook

1.1 General introduction

1.1.1 Thesis outline

The global asymptotic stabilization of nonlinear control systems is an important field in control theory, particularly for mathematical models of population dynamics. Over the last century, many control tools have been developed to design stabilizing feedback laws for control systems. Examples include control Lyapunov functions, damping, homogeneity, averaging, backstepping, forwarding, and transverse functions (see [44, Chapter 12]). However, these strategies do not ensure the positivity of the control function, limiting their applicability in many cases and, in particular when the control function is a density of release.

In this work, we consider a control system that models the management of vector populations, particularly mosquito populations, using the release of sterile male insects. These sterile males compete with wild males to fertilize the females. Globally and asymptotically stabilizing such a control system has the advantage of reducing release costs when the population approaches zero. The primary challenge of this process is its robustness against dynamic parameter or population perturbations (which can be due to, for instance, the arrival of mosquitoes transported by human activities), which can cause population resurgence. This challenge can be addressed by finding global stabilizing feedback laws for the dynamics to zero.

This thesis focuses on the asymptotic stabilization of various mathematical models related to the sterile insect technique (SIT) using a feedback control strategy.

Our research is divided into three parts. Part II concerns the feedback stabilization of the SIT ODE model (see Chapter 2, Chapter 3 and Chapter 5). Part III explores the extension of control feedback design tools, particularly reinforcement learning (RL) (see Chapter 4). Part IV focuses on the mathematical analysis of a PDE model of the SIT with an Allee effect in a two-dimensional bounded domain with Neumann boundary conditions and its global asymptotic stability via a feedback law (see Chapter 6).

1.1.2 Mosquito population

Mosquitoes are an integral part of natural ecosystems in most wetlands (see Figure 1.1). There are numerous families of species worldwide, but the most common are *Aedes*, *Culex*, and *Anopheles*.

Culex mosquitoes usually bite at night, both indoors and outdoors. Although they prefer avian species, they can also bite humans. These mosquitoes are vectors for diseases such as the West Nile virus and Eastern/Western equine encephalitis. They lay their eggs on the surface of freshwater sites and in artificial containers.

Anopheles mosquitoes bite both indoors and outdoors, primarily from dusk to dawn. They prefer humans and mammals as hosts and are predominantly found in Africa. They are the primary vectors of malaria. Their eggs float on the surface of natural water bodies with vegetation, such as ponds and swamps.

Aedes mosquitoes prefer to bite during the day, primarily targeting humans. They are vectors of diseases such as the Zika virus, chikungunya, yellow fever, and dengue fever.

In tropical and equatorial zones, mosquitoes are present nearly year-round in both their larval and adult stages. However, as one moves closer to the poles, mosquito populations become more seasonal, with a significant time lag between egg-laying and the emergence of larvae and adults in the winter on at certain periods.

Among mosquito-borne diseases, malaria is the most harmful. According to the World Health Organization (WHO), there were 263 million cases of malaria in 2023 compared to 252 million cases in 2022. The estimated number of malaria-related deaths stood at 597,000 in 2023 (see [85]). The African continent is the most affected, particularly among children under five years old (see Figure 1.2).

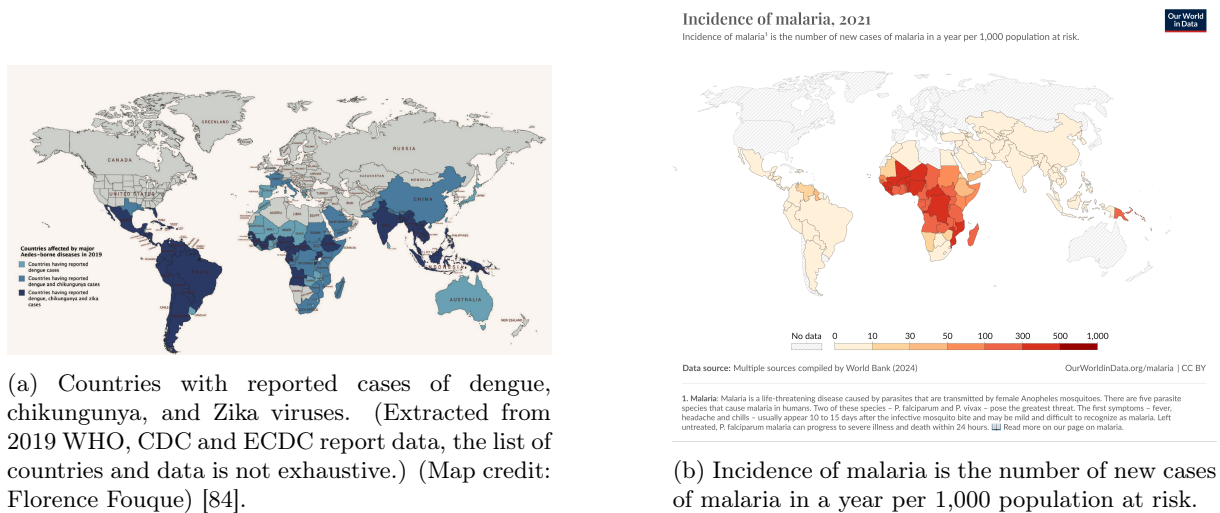


Figure 1.1

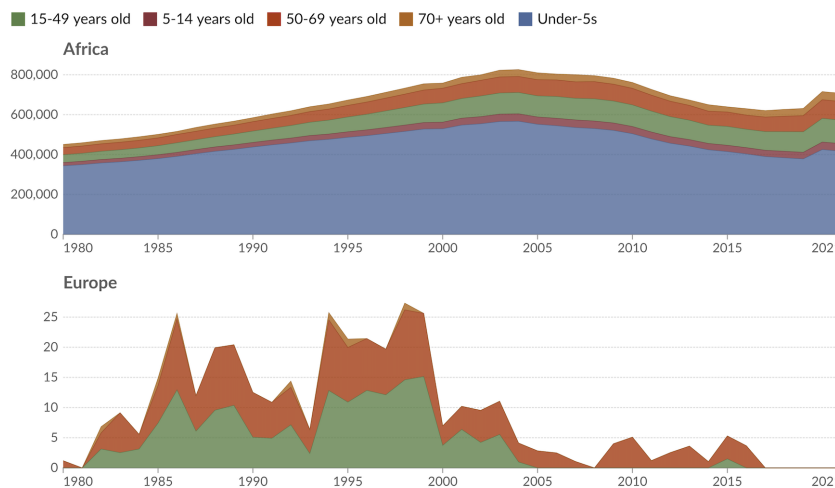


Figure 1.2: Malaria deaths by age [92]

Anopheles was eradicated in Brazil in 1940 [42] and in Egypt in 1945 [96] through the application of residual insecticide on walls and ceilings, which prevented female mosquitoes from approaching habitats and biting the population. The use of DDT increased the effectiveness of residual treatments and helped stop the progress of malaria in Taiwan, the USA, and Italy. Consequently, the World Health Assembly launched the Global Malaria Eradication Program with the application of DDT in 1955 [112].

This program successfully eradicated malaria in temperate regions of Europe and Asia, in some tropical areas of the Mediterranean Basin and the southern USA, and on several tropical islands of the Caribbean [53]. Malaria was also significantly reduced in Brazil and India. Spectacular progress was made in Sri Lanka, where the number of cases was reduced from over two million annually in 1958 to just 17 cases in 1963 [66].

However, after spraying was halted and the consolidation phase of the program was implemented, some areas experienced a resurgence of mosquito populations due to hydraulic or agricultural projects that increased natural habitats, as well as the development of resistance to DDT. Consequently, malaria incidence rose again in these regions [66]. In 1967, the World Health Assembly discontinued the eradication program, and sub-Saharan Africa failed to benefit from the Global Malaria Eradication Program (1955–1967).

Following this program, the WHO conducted a pilot study on a residual treatment program combined with the administration of anti-malarial drugs in the Garki district of northern Nigeria between 1969 and 1976. This study demonstrated that in areas where ecological conditions favored extremely high vector populations, the combination of residual treatment and drug administration was insufficient to halt malaria transmission [66]. Moreover, the high toxicity of DDT to humans was discovered [107], leading to its ban in most countries during

the 1970s.

Aedes mosquitoes are highly adaptable, requiring only water, flowers, and blood for their development. As a result, areas that were once free of mosquitoes are now under threat. *Aedes albopictus* is colonizing certain European countries, notably France and Italy. With global warming, international trade and population movements, this process is being accelerated (see 1.3).

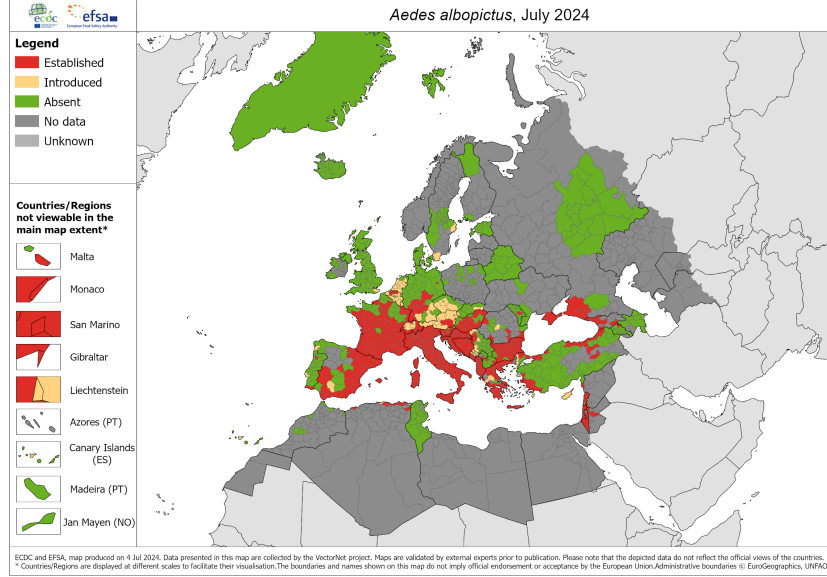


Figure 1.3: *Aedes albopictus* - current known distribution of July 2024 in Europe [113]

In ecosystems, mosquitoes contribute to the carbon and nitrogen cycles and serve as bioindicators. They also play a role in pollination. Consequently in areas where they are not an invasive species, radical measures cannot be adopted to reduce their population (to contribute to the fight against mosquito-borne diseases) without risking disruption of the ecological balance. This is why combating vector-borne diseases transmitted by mosquitoes is very challenging, particularly regarding the strategies employed to reduce mosquito vectors.

In addition to biological strategies, such as encouraging their natural predators like dragonflies, fish and reptiles, the Sterile Insect Technique (SIT) is increasingly being utilized in this context. It is used to combat *Aedes* species in countries such as Brazil, Cuba, Malaysia, Mexico, the USA, France (Reunion Island), Germany, Italy, Greece, and Spain [84].

In addition to the traditional tools used in mosquito control, mathematical research, particularly in population dynamics, contributes to the monitoring and management of mosquito populations established in a given area. Referring to the life cycle of mosquitoes, for any time $t \geq 0$, we denote by $E(t) \geq 0$ the density of mosquitoes in their aquatic phase (although, to simplify, we will often just call them "eggs" in this work), by $F(t) \geq 0$ the density of adult females, and by $M(t) \geq 0$ the density of adult males. The carrying capacity is denoted by K . It can be interpreted as the maximum density of eggs that females can lay in breeding sites. Since here the larval and pupal compartments are not present, it is as if E represents all the aquatic compartments, in which case in this term K represents a logistic law's carrying capacity for the aquatic phase that also includes the effects of competition between larvae. For all $t \geq 0$ the mosquito population mathematical model is

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) \Lambda(M) - (\nu_E + \delta_E) E, \\ \dot{M} = (1 - \nu) \nu_E E - \delta_M M, \\ \dot{F} = \nu \nu_E E - \delta_F F, \end{cases} \quad (1.1)$$

where, $\beta_E > 0$ is the oviposition rate; $\delta_E, \delta_M, \delta_F > 0$ are the death rates for eggs, wild adult males and females respectively; $\nu_E > 0$ is the hatching rate for eggs; $\nu \in (0, 1)$ is the probability that a pupa gives rise to a female, and $(1 - \nu)$ is the probability that it gives rise to a male. The term $\Lambda(M)$ represents how the density of males influences the dynamics of the aquatic phase through mating and $\Lambda(M) = 1$ means that every female is inseminated by a male and that all eggs are viable which is a reasonable assumption for a high-density population. Indeed, in their life cycle, male mosquitoes mature before female mosquitoes for *Aedes* mosquitoes. As a result in high population settings, most female mosquitoes are fertilized before leaving their spawning sites. In tropical

or equatorial zones, the Allee effect can therefore be neglected in normal conditions. The basic offspring number for the corresponding system is given by:

$$\mathcal{R}_0 = \frac{\beta_E \nu \nu_E}{\delta_F(\nu_E + \delta_E)}. \quad (1.2)$$

$\mathcal{R}_0$ represents the number of adult females produced by one adult female during her lifespan. When its value is smaller than one, the mosquito population goes extinct; otherwise, the population persists. The Allee term considered in this study is

$$\Lambda(M) = \frac{\eta M}{1 + \eta M}. \quad (1.3)$$

This Allee effect term takes into account the difficulty of the females to find a male with whom to mate when the density of the population is low. η is the search efficiency parameter and $1/\eta$ corresponding the constant male density for which the mating probability of a female would be $1/2$ (see [49] where the notation is $\theta = 1/\eta$). As described in [49] (where the previous form is called the rectangular hyperbola function), other forms can be used for representing this Allee effect but, for simplicity, we chose to use this basic one in this work. A comparison of the phase portraits of F and M when E is at equilibrium shows that this model and the one in [102] where $\Lambda(M) = 1 - \exp(-\eta M)$ are bistable and share nearly the same basin of attraction. However, from a mathematical perspective, the Allee term $\eta M/(1 + \eta M)$ appears to simplify the analytical study. The basic offspring number of the corresponding model is

$$\mathcal{R}_0 = \frac{\beta_E \nu \nu_E}{\delta_F(\nu_E + \delta_E)} - \frac{2\delta_M}{\eta K(1 - \nu)\nu_E} \left(1 + \sqrt{1 + \frac{\eta K(1 - \nu)\nu_E}{\delta_M}} \right). \quad (1.4)$$

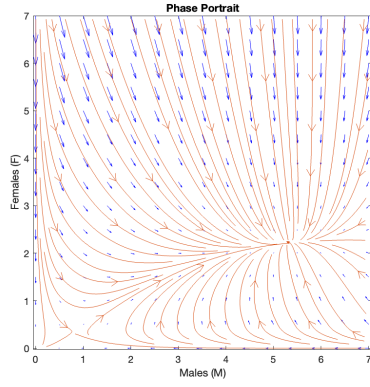
The mathematical properties of the model incorporating the Allee term $1 - \exp(-\eta M)$ have already been investigated in [102]. Figure 1.4 shows the phase portrait of F and M for the SIT model for each Allee term. Figure 1.4 (a) and (b) show that the phase portrait are approximately the same, despite the difference between the Allee term function shown in Figure 1.4(c).

1.1.3 Sterile Insect Technique

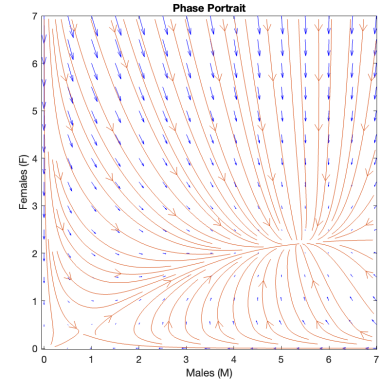
Bushland's research (1951–1953) shows that exposing screw-worm larvae to X-rays or gamma rays renders the males sexually sterile without impairing too much their mating behavior. His studies revealed that females that mate with irradiated males produce eggs that fail to hatch. Based on this experimental evidence, Edward F. Knippling (see [67]) proposed the hypothesis that if sterile males are initially released at a density high enough to reduce the wild population, then maintaining the release of a sufficiently big number of sterile males each generation will progressively suppress the wild population to an even greater degree with each successive generation. SIT successfully eradicated the screwworm fly in Curacao [24] and quickly was extended to the mosquito population with the largest-scale trial projects in El Salvador and India [66]. The Sterile Insect Technique (SIT), when applied to mosquito control, offers the advantage of reducing vector populations with a reversible effect on the ecosystem. This is not the case with methods such as the *Wolbachia* technique, which replaces mosquito populations with non-disease-transmitting mosquitoes, resulting in irreversible ecological effects. Furthermore, SIT helps reduce mosquito bites, which are a nuisance to affected populations.

The success of the SIT, is essentially based on the production of sterile males of high quality (the sterilization process must not impair significantly their ability to survive, fly, disperse or mate compared to wild males), in sufficient quantity and with an appropriate release strategy. To enhance the quality of sterile male mosquitoes, additional methods are employed, such as treating sterile males with chemical pheromones to increase their sexual attractiveness compared to wild males. The primary focus of this thesis, however, is on the quantity of sterile males needed and the strategies for their release.

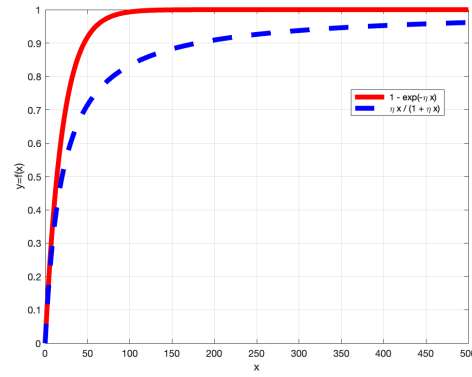
We present the mathematical modeling of mosquito population density dynamics when sterile males are released. Let $M_s(t) \geq 0$ for all $t \geq 0$ represent the density of sterile male mosquitoes, and $u(t) \geq 0$ the density of sterile males released into the population. By adding the probability of a female to mate a wild male to the evolution equation for females, F models the density of females inseminated by wild males. To simplify we also drop the Allee term in the equation aquatic phase. The Sterile Insect Technique (SIT) model is then given as follows, for all $t \geq 0$:



(a) Phase portrait of F and M for SIT model system Allee effect term $\eta M/(1 + \eta M)$ for $\eta = 0.05$.



(b) Phase portrait of F and M for SIT system with Allee effect term $1 - \exp(-\eta M)$ for $\eta = 0.05$.



(c) Allee effect terms for $\eta = 0.05$.

Figure 1.4

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \\ \dot{F} = \nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F, \\ \dot{M} = (1 - \nu) \nu_E E - \delta_M M, \\ \dot{M}_s = u(t) - \delta_s M_s. \end{cases} \quad (1.5)$$

Here, $\delta_s > 0$ is the death rate of sterile males, and $0 < \gamma_s \leq 1$ represents the mating competitiveness of M_s . In this form, F represents the density of female mosquitoes mating with wild males, while it is assumed that all other females automatically mate with sterile males. The previous model can be extended by incorporating the young female stage. The system becomes :

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\delta_E + \nu_E)E, \\ \dot{M} = (1 - \nu)\nu_E E - \delta_M M, \\ \dot{Y} = \nu\nu_E E - \frac{\eta_1 M}{M + M_s} Y - \frac{\eta_2 M_s(t)}{M + M_s} Y - \delta_Y Y, \\ \dot{F} = \frac{\eta_1 M}{M + M_s} Y - \delta_F F, \\ \dot{U} = \frac{\eta_2 M_s}{M + M_s} Y - \delta_U U, \\ \dot{M}_s = u(t) - \delta_s M_s. \end{cases} \quad (1.6)$$

In this model, the compartment Y represents the density of young females (those that have not yet started their gonotrophic cycle), while U denotes the density of females that mate with sterile males. Monitoring the evolution of U and F is particularly important for understanding disease transmission dynamics, as females in both compartments require blood meals. We assume that a female mating mosquito has probability $\frac{M}{M+M_s}$ to mate with a wild male and probability $\frac{M_s}{M+M_s}$ to mate with a sterile one. Hence, the transfer rate from the compartment Y splits into transfer rate of $\frac{\eta_1 M}{M+M_s}$ to compartment F and a transfer rate of $\frac{\eta_2 M_s}{M+M_s}$ to compartment U of females that will be laying sterile (nonhatching) eggs. η_1 and η_2 describe preference that female may have between wild and sterile males.

In the field of release strategies, aerial tools, notably drones, have proven to be highly useful, for releasing sterile insects. Such release systems have been developed for SIT programs targeting fruit flies, moths, and tsetse flies [24]. Similarly, in [105], a comparable system was designed for the release of adult sterile male *Aedes* mosquitoes. According to [105], this system can be calibrated to fix the total number of sterile insects released per unit area. This should ensure a more precise release, accounting for the natural dispersion of insects in the field, and allow for the coverage of large intervention areas. For mathematical monitoring of the population in this framework, we propose in this study to incorporate the two-dimensional dispersion of the mosquito population within a bounded domain under the release of sterile males. For the sake of simplicity, we will assume that Ω is regular enough and that all biological parameters remain constant over time, thus disregarding the effects of field heterogeneity and seasonal variations, except for the carrying capacity K . We assume that $K \in \mathcal{C}^0(\bar{\Omega}; (0, +\infty))$. Let $(E^0, F^0, M^0)^T : \Omega \rightarrow [0, +\infty)^3$ be the mosquito population at the initial time $t = 0$. Adopting the same approach as in (1.1), the mosquito population density $(E(t, x), F(t, x), M(t, x))^T$ at time $t \geq 0$ and position $x \in \Omega$ is the solution of the Cauchy problem

$$\begin{cases} \frac{\partial E}{\partial t} = \beta_E F \left(1 - \frac{E}{K(x)}\right) \frac{\eta M}{1 + \eta M} - (\nu_E + \delta_E)E, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial F}{\partial t} - d_1 \Delta F = \nu\nu_E E - \delta_F F, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial M}{\partial t} - d_2 \Delta M = \nu\nu_E E - \delta_M M, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial F}{\partial n} = \frac{\partial M}{\partial n} = 0, \quad t \geq 0, \quad x \in \partial\Omega, \\ (E(0, x), F(0, x), M(0, x))^T = (E^0(x), F^0(x), M^0(x))^T, \quad x \in \Omega, \end{cases} \quad (1.7)$$

where n is the outward unit normal to Ω . The SIT model in this setting becomes:

$$\begin{cases} \frac{\partial E}{\partial t} = \beta_E F \left(1 - \frac{E}{K(x)}\right) \frac{\eta M}{1 + \eta(M + \gamma M_s)} - (\nu_E + \delta_E)E, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial F}{\partial t} - d_1 \Delta F = \nu\nu_E E - \delta_F F, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial M}{\partial t} - d_2 \Delta M = (1 - \nu)\nu_E E - \delta_M M, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial M_s}{\partial t} - d_3 \Delta M_s = u - \delta_s M_s, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial F}{\partial n} = \frac{\partial M}{\partial n} = \frac{\partial M_s}{\partial n} = 0, \quad t \geq 0, \quad x \in \partial\Omega, \end{cases} \quad (1.8)$$

Remark that the presence of sterile males is taken into account in the Allee term of the aquatic phase equation and that in (1.7) and (1.8) F represents again all the females as in (1.1).

1.2 Main results

Let us first present the main challenge when applying the sterile insect technique. We consider the following result from [12], which pertains to constant control applied to system (1.5). We define

$$U^* = \mathcal{R}_0 \frac{K(1-\nu)\nu_E\delta_s}{4\gamma_s\delta_M} \left(1 - \frac{1}{\mathcal{R}_0}\right)^2. \quad (1.9)$$

If $u(\cdot)$ denotes a constant control function equal to some $\bar{U} > U^*$ for all $t \geq 0$, then the corresponding solution $(E(t), M(t), F(t), M_s(t))$ converges to $(0, 0, 0, \bar{U}/\delta_s)$ as $t \rightarrow \infty$.

In practice, after the population decline caused by this intervention, it is unrealistic to maintain this release indefinitely. For instance, we choose $\bar{U} := 2U^*$ and run the system as long as $E(t) > 0.1$ with $u(t) = \bar{U}$, i.e., the control function (1.10):

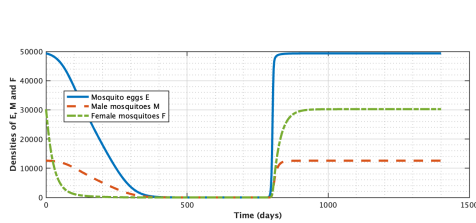
$$u(t) = \begin{cases} \bar{U}, & \text{as long as } E(t) > 0.1, \\ 0, & \text{from there on.} \end{cases} \quad (1.10)$$

Figure 1.5(a) illustrates the resurgence of the population when the control is stopped. We then deduce that a minimal release is required if we want to avoid population resurgence. What is the size of this minimal release?

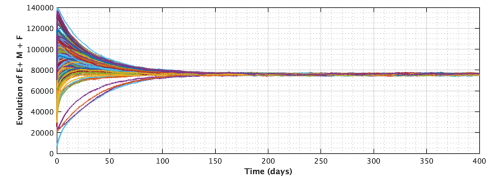
We analyze the robustness of this control through the following protocol: the constant control law U^* given by (1.9) is computed using fixed parameter values corresponding to those chosen in Table 4.5. However, for computing the real dynamics of system (1.7), we introduce simultaneous random perturbations in the system parameters, following the distributions:

$$\begin{aligned} \hat{\beta}_E &\sim \mathcal{U}(7.46, 14.85), \\ \hat{\nu}_E &\sim \mathcal{U}(0.005, 0.25), \\ \hat{\delta}_E &\sim \mathcal{U}(0.023, 0.046), \\ \hat{\delta}_F &\sim \mathcal{U}(0.033, 0.046), \\ \hat{\delta}_M &\sim \mathcal{U}(0.077, 0.139), \end{aligned} \quad (1.11)$$

where $\mathcal{U}(a, b)$ denotes the uniform distribution on the interval $[a, b]$. Figure 1.5 shows 200 simulations with random initial conditions in $[0, 10K]^4$.



(a) Evolution of the densities of E , F , and M under (1.10).



(b) Evolution of $E + F + M$ for 200 random initial conditions when $u(t) = \bar{U}$ and the parameters given in Table 4.5. The mosquito dynamics are simulated with these parameters, randomly sampled for each simulation according to (1.11).

Figure 1.5

Although $\lim_{t \rightarrow +\infty} u(t) = \bar{U}/\delta_s$, Figure 1.5(b) highlights the failure of constant control in asymptotically stabilizing the system when parameters are perturbed. Given the population decline caused by prior interventions, what density of sterile males should we release to maintain the population closer to zero despite disturbances in population life parameters?

1.2.1 Feedback control design of ODE SIT models

Nonlinear control laws

In control theory, several techniques exist that allow for the analytical construction of control laws. In our case, the structure of the dynamics (1.5) is particularly advantageous, as it can be decomposed into two subsystems:

the interdependent subsystem formed by $z = (E, M, F)$, and the M_s subsystem, which evolves independently of the other variables.

$$\begin{cases} \dot{z} = f(z, M_s), \\ \dot{M}_s = u - \delta_s M_s, \end{cases} \quad (1.12)$$

This particular structure motivates the use of the *Backstepping* technique to construct the control law. In the general context of applications, this technique is better suited when the control is not subject to constraints. However, in our case, the control law we seek must remain positive, as we are working in a biological setting where the control represents the density of sterile mosquitoes to be released into the target population. By considering the dynamics of the equations for E , M , and F in the system (1.5), with M_s as the control input, in accordance with the Backstepping method presented in [44, Theorem 12.24], we observe that setting $M_s = \theta M$ simplifies the expression $\frac{M}{M + \gamma_s M_s}$. Moreover, this choice has a biologically meaningful interpretation: it implies that the ratio of released sterile male mosquitoes to wild male mosquitoes is equal to θ . The asymptotic stability of this subsystem

$$\dot{z} = f(z, \theta M) = \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E \\ (1 - \nu) \nu_E E - \delta_M M \\ \frac{\nu \nu_E}{1 + \gamma_s \theta} E - \delta_F F \end{pmatrix}. \quad (1.13)$$

can be fully analyzed. We establish that stability holds if and only if $\theta > \theta^*$, where θ^* is an explicitly determined constant. When this condition is satisfied, i.e., $\theta > \theta^*$, we construct an explicit Lyapunov function for the three-dimensional closed-loop system. This construction is a crucial step for applying the backstepping method. Our Lyapunov function is homogeneous of degree one, which is physically more meaningful than a conventional quadratic Lyapunov function. To design a control law u that both satisfies the positivity constraint and stabilizes the four-dimensional system (1.12), we penalize deviations from the relation $M_s \neq \theta M$. Rather than employing the classical penalty term $(M_s - \theta M)^2/2$ to be added to the Lyapunov function, we introduce a modified penalty of the form $\alpha(M_s - \theta M)^2/(\theta M + M_s)$, which has the advantage to be also homogeneous of degree one. This adjustment enables the derivation of a control u that remains non-negative and ensures global stabilization of system (1.12). The same methodology is employed in the analysis of model (1.6)

We study the stabilizing control laws for both systems by assuming that $\delta_s > \delta_M$. Indeed, the sterilizing and the release process can reduce the longevity of the sterile male mosquito. This is also the difficult case since if $\delta_s \leq \delta_M$ the sterile mosquitoes last longer and it is easier to implement SIT. Theorem 2.2.1 proves that the basic offspring number for the uncontrolled system (1.1) for $\Lambda(M) = 1$, is given by:

$$\mathcal{R}_0 = \frac{\beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E)}. \quad (1.14)$$

When $\mathcal{R}_0$ is smaller than one, the mosquito population goes extinct; otherwise, the population persists. In Chapter 2 of this thesis, we show that when the density of the sterile mosquitoes is proportional to the density of wild male mosquitoes with a coefficient θ , the basic offspring number of the SIT dynamics (1.5) can be bounded by:

$$\mathcal{R}(\theta) = \frac{\mathcal{R}_0}{1 + \gamma_s \theta}, \quad (1.15)$$

Thanks to the backstepping method, we construct the feedback law given by

$$u_{\theta, \alpha}(E, M, F, M_s) := \max(0, G(E, M, F, M_s)), \quad (1.16)$$

where for any $\alpha > 0$ and $\beta_s > 0$ and

$$\psi := \frac{2\beta_E \nu \nu_E}{\delta_F (1 - \mathcal{R}(\theta))(1 + \gamma_s \theta)}, \quad (1.17)$$

G is defined by

$$\begin{aligned} G(E, M, F, M_s) := & \frac{\gamma_s \psi E (\theta M + M_s)^2}{\alpha (M + \gamma_s M_s) (3\theta M + M_s)} \\ & + \frac{((1 - \nu) \nu_E \theta E - \theta \delta_M M) (\theta M + 3M_s)}{3\theta M + M_s} \\ & + \delta_s M_s + \frac{\beta_s}{\alpha} (\theta M - M_s), \text{ if } M + M_s \neq 0, \end{aligned} \quad (1.18)$$

$$G(E, M, F, M_s) := 0, \text{ if } M + M_s = 0. \quad (1.19)$$

We prove in Theorem 2.3.2 that for any $\alpha > 0$, when applying this feedback control law for $\theta > \frac{\mathcal{R}_0 - 1}{\gamma_s}$, then $\mathbf{0}$ is globally exponentially stable for the system (1.5) with a convergence rate at least equal to c given by:

$$c = \min \left\{ \frac{\beta_s}{\alpha}, \frac{\nu \nu_E (1 - \mathcal{R}(\theta))}{1 + \mathcal{R}(\theta)}, \delta_M, \frac{\beta_E \delta_F (1 - \mathcal{R}(\theta))}{2} \right\}. \quad (1.20)$$

This feedback control $(E, M, F, M_s)^T \mapsto u_{\theta, \alpha}(E, M, F, M_s)$ is robust for perturbation (the test is carried out in [2]) of the dynamics parameters and satisfies

$$\lim_{|(E, M, F, M_s)| \rightarrow 0} u_{\theta, \alpha}(E, M, F, M_s) = 0. \quad (1.21)$$

This feedback law ensures that a sufficient quantity is released zero to prevent population emergence. It is important to note that this feedback is independent of the carrying capacity. Figure 1.2.1 shows 200 simulations with random initial conditions in $[0, 10K]^4$.

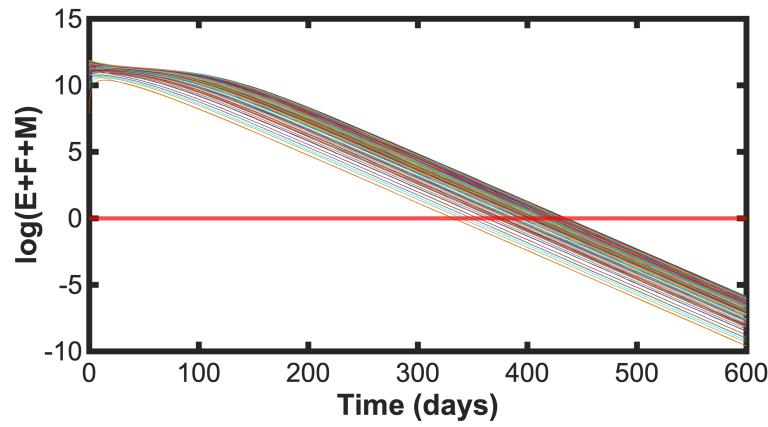


Figure 1.2.1. Evolution of $\log(E + F + M)$ for 200 random initial conditions when applying feedback (2.94) computed with $\theta = 220$, $\alpha = 13$, $\beta_s = 1 \text{ Day}^{-1}$ and the parameters given in table 4.5, which are not the parameters used for simulating the mosquito dynamics (the latter being taken randomly for each simulation according to (2.115))

These works were carried out in the first chapter of the thesis. However, the field application led us to question the actual availability of the various measurements required for the implementation of this closed-loop control. Following the discussions we had, on the one hand with René Gato, and on the other hand with Hervé Bossin, we learned that the most easily measurable states are M , M_s , and $M + M_s$. We therefore undertook to construct a *feedback* control law that depends only on these states. We established some results for linear controls of the form $u = \lambda M$ and $u = \alpha(M + M_s)$, which we present later. In the second chapter of the thesis, we use another tool from control theory: the construction of observers, which allows the estimation of the full state of a dynamical system from the measurement of only some of its states. In this chapter, we consider the model (1.6), which is more complete than the one studied in Chapter 2. This model is six-dimensional. First, we construct a full state feedback law that stabilizes (1.6). Then, assuming that the measured states are M and M_s , we estimate the remaining four states using an observer dynamic.

Theorem 3.3.1 proves that the basic offspring number for system (1.6) is

$$\mathcal{R}_0^Y = \frac{\eta_1 \beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\eta_1 + \delta_Y)} \quad (1.22)$$

In Chapter 3, we show that when the density of the sterile mosquitoes is proportional to the density of wild male mosquitoes with a coefficient θ , the basic offspring number of the SIT dynamics (1.6) can be bounded by:

$$\mathcal{R}^Y(\theta) := \frac{\beta_E \eta_1 \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\Delta \eta + (1 + \theta)(\eta_2 + \delta_Y))}. \quad (1.23)$$

Thanks to the backstepping method one has the following result presented in Chapter 3.

We define

$$\phi := \frac{(2 + \mathcal{R}(\theta))\eta_1\beta_E\nu\nu_E - 3\mathcal{R}(\theta)\Delta\eta\delta_F(\nu_E + \delta_E)}{\delta_F(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))(1 + \theta)} - \frac{\eta_1\beta_E\nu\nu_E\mathcal{R}(\theta)}{(1 + \theta)^2(\delta_E + \nu_E)\delta_F}, \quad (1.24)$$

$$Q := 3(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E)\delta_F - (1 - \mathcal{R}(\theta))\eta_1\beta_E\nu\nu_E, \quad (1.25)$$

and for $\alpha > 0$, we define the map $G : \mathcal{N} := [0, +\infty)^6 \rightarrow \mathbb{R}$, $(E, Y, F, U, M, M_s)^T \mapsto \mathcal{G}(E, Y, F, U, M, M_s)$ by

$$\begin{aligned} \mathcal{G}(E, Y, F, U, M, M_s) := & \frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)(3\theta M + M_s)} + \frac{((1 - \nu)\nu_E\theta E - \theta\delta_M M)(\theta M + 3M_s)}{3\theta M + M_s} \\ & + \delta_s M_s + \frac{1}{\alpha}(\theta M - M_s) \text{ if } M + M_s \neq 0, \end{aligned} \quad (1.26)$$

$$\mathcal{G}(E, Y, F, U, M, M_s) := 0 \text{ if } M + M_s = 0. \quad (1.27)$$

Finally,

$$\mathcal{U}_{\theta, \alpha}(E, Y, F, U, M, M_s) := \max(0, \mathcal{G}(E, Y, F, U, M, M_s)). \quad (1.28)$$

The global stability result is the following (see Theorem Proposition 3.3.1 below).

Theorem 1.2.1. *Assume that $\mathcal{R}^Y(\theta) < 1$. Then $\mathbf{0} \in \mathcal{N}$ is globally exponentially stable in $\mathcal{N}$ for system (1.6) with the feedback law (1.28). The exponential convergence rate is bounded by the positive constant c_p defined by*

$$c_p := \min\left\{c, \frac{1}{\alpha}, \delta_M, \frac{Q}{3(1 + \theta)\delta_F(\nu_E + \delta_E)}, \delta_U\right\}. \quad (1.29)$$

The advantage of this dynamics is that it is more realistic than (1.5). One can also measure the decreasing rate of both kinds of female in the model which can be useful in epidemiology as both types transmit the disease. The main challenge for applying feedback laws in general is the measurement of the dynamic states. This issue is addressed in Chapter 3 for the complete system (1.6), where an observer is constructed to estimate the states of the dynamical system (1.6) using only measurements of wild and sterile males. For nonlinear systems, there is no general formula for constructing an observer. In our case, considering that the measured states are sterile males and wild males, the output dynamics exhibits a particular structure given by (1.6) as

$$\begin{cases} \dot{X} = AX + B(y)X + Du, \\ y = CX, \end{cases} \quad (1.30)$$

where the pair (A, C) is observable, and that all components of the matrix $B(y)$ are bounded. A state observer corresponding to (1.30) is given as follows:

$$\begin{cases} \dot{\hat{X}}(t) = A\hat{X}(t) + B(y(t))\hat{X} + Du(t) - L(C\hat{X} - y(t)), \\ \hat{y}(t) = C\hat{X}(t), \end{cases} \quad (1.31)$$

The observation problem consists in finding a gain L such that the dynamics of observer error $\hat{X}(t) - X(t)$ converges exponentially towards zero. We use the following results in [115].

Theorem 1.2.2. *The observer error converges exponentially towards zero if there exist matrices $P = P^T > 0$ and R of appropriate dimensions such that following Linear Matrix Inequalities (LMIs) are feasible:*

$$\mathcal{A}^T(\eta)P - C^T R + P\mathcal{A}(\eta) - R^T C + \xi I < 0, \quad (1.32)$$

$$\forall \eta \in \mathcal{V}_{S_{n,n}}, \quad (1.33)$$

for some constant $\xi > 0$. When these LMIs are feasible, the observer gain L is given by $L = P^{-1}R^T$.

We prove that applying the output feedback law with the estimated states also stabilizes the dynamics to zero. This state observer is used to estimate both eggs E and young females Y . Here we show that $\mathbf{u}(\hat{X}, y)$ stabilizes the dynamics at the origin. We consider the coupled system

$$\begin{cases} \dot{X} = f(X, \hat{\mathbf{u}}(\hat{X}, y)), \\ \dot{\hat{X}} = f(\hat{X}, \hat{\mathbf{u}}(\hat{X}, y)) - L(C\hat{X} - y), \end{cases} \quad (1.34)$$

with

$$\hat{\mathbf{u}}(\hat{X}, y) = \max \left(0, S(\hat{X}, y) \right). \quad (1.35)$$

where $S : \mathbb{R}^4 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$, $(\hat{X}, y)^T \mapsto S(\hat{X}, y)$ is defined by

$$S(\hat{X}, y) := G(\hat{E}, M, \hat{Y}, M_s) \quad (1.36)$$

One then have the following theorem (see Theorem 3.5.1)

Theorem 1.2.3. *Assume that $\mathcal{R}^Y(\theta) < 1$. Then $\mathcal{O} \in \mathcal{E} = \mathcal{N} \times \mathbb{R}^6$ is globally exponentially stable in $\mathcal{E}$ for system (1.34) with the feedback law (1.35). The convergence rate is bounded by the positive constant c_e defined by*

$$c_e := \min \{ c_1, c_2, c', \frac{\xi}{4} \}. \quad (1.37)$$

This observer can be employed to any stabilizing feedback law and its main advantage is the reduction of the measuring cost. One need only to measure wild and sterile males to estimate the other components of the states of the system.

Linear control laws

Some simple linear feedback laws are also tested for the system (1.5) in Chapter 2, such as feedback laws that depend linearly on the number of wild male mosquitoes ($u(x) = \lambda M$) or the total number of males (both wild and sterile, $u(x) = k(M + M_s)$). Measuring the total number of males is easier than measuring only wild males. However, the feedback law that depends linearly on wild males is more robust than the one that depends linearly on the total number of males. The main challenge of such a feedback is the stability proof. It is easy to prove that under some assumption ((2.125) on k and (2.189) on λ), the only equilibrium of the system is zero. So one can conjecture the globally asymptotical stability of such an equilibrium. We prove in Chapter 2 that under these assumption one has some invariant set in which every solution converges to zero asymptotically. The proof of global attraction of such an invariant set remains an open question.

Despite the non-linearity of the backstepping control, it offers, thanks to the regulation parameters α and θ , more flexibility and better results in terms of both convergence time and control cost.

Moreover we show in Chapter 5 that when females have no preference for fertile males (i.e. $\gamma = 1$), the feedback laws (1.38) and (1.39) globally asymptotically stabilize the population to zero

$$u(x) = 2\mathcal{R}_0(\delta_s - \delta_M)M + \mathcal{R}_0(1 - \nu)\nu_E E, \quad (1.38)$$

$$u(x) = (\delta_s - \delta_M)(M + M_s) + \mathcal{R}_0(1 - \nu)\nu_E E, \quad (1.39)$$

when their death rate is higher than the death rate of wild males and where $\mathcal{R}_0$ is the basic offspring number of the target insect population. In contrast to the linear control laws $u = k(M + M_s)$ and $u = \lambda M$, for which convergence zero could not be established, we were able to prove the convergence of the closed-loop system to zero.. Our approach is based on bounding the ratio $\frac{M}{M + \gamma M_s}$, where M and M_s are the solutions of the closed-loop system, as well as on the exponential convergence of the subsystem (E, F, M) of the same system to zero.

Construction of control by RL

Up to now, we used analytical methods to design a feedback control law that globally stabilizes the control system to zero. The difficulty of obtaining the required measurements is addressed by an observer system that provides an estimate of all the dynamic states based on sterile and wild male measurements. Among the analytical tools for feedback control design, none provides a feedback law that immediately depends on the system output. For instance, in the case of mosquito population management through sterile male release, the most accessible output states are the total female population (including young females, females that mate with wild males and those that mate with sterile males) and the total male population (sterile plus wild males).

Is it possible to construct a feedback law that directly depends on these measurements? There is no general answer to this question for nonlinear output control systems. Here, we propose a numerical solution of this problem using Reinforcement Learning (RL). RL is a subfield of machine learning concerned with training *agents* to make decisions in an environment to maximize a long-term *reward* function. From a control perspective, this can equivalently be seen as finding an optimal control for a cost function over the trajectories. The goal is to blend the RL methodology with mathematical analysis to extract an explicit feedback control depending on easier measurements, notably the total adult female and total adult male populations. Our approach is summarized in Fig. 1.6. We start by discretizing our system of equations (1.5), and use those dynamics to create a simulation of our model. We use it to create an environment by implementing the observations, actions and rewards. Using this

code, we train an RL agent that learns to maximize the objective function we assign it through many simulations, using the PPO algorithm. Once the policy has converged, we can evaluate it on any simulation, in particular we can query a control u for any current state $(E, M, F, M_s)^T$. Since we only trained policies with 1 or 2 observations, we can plot the action as a function of the input in 1D or 2D space. This allows us to perform a regression and empirically write a simple explicit control that has the same general shape as the neural network control. This leads to the following simplified formula for the feedback control:

$$v_{\text{reg}}(M + M_s, F + F_s) = \begin{cases} u_{\min} & \text{if } \log \frac{M+M_s}{F+F_s} > \alpha_2, \\ u_{\max} & \text{otherwise,} \end{cases} \quad (1.40)$$

for $M + M_s, F + F_s > 0$, where $\alpha_2 = 4$ is a constant found by regression.

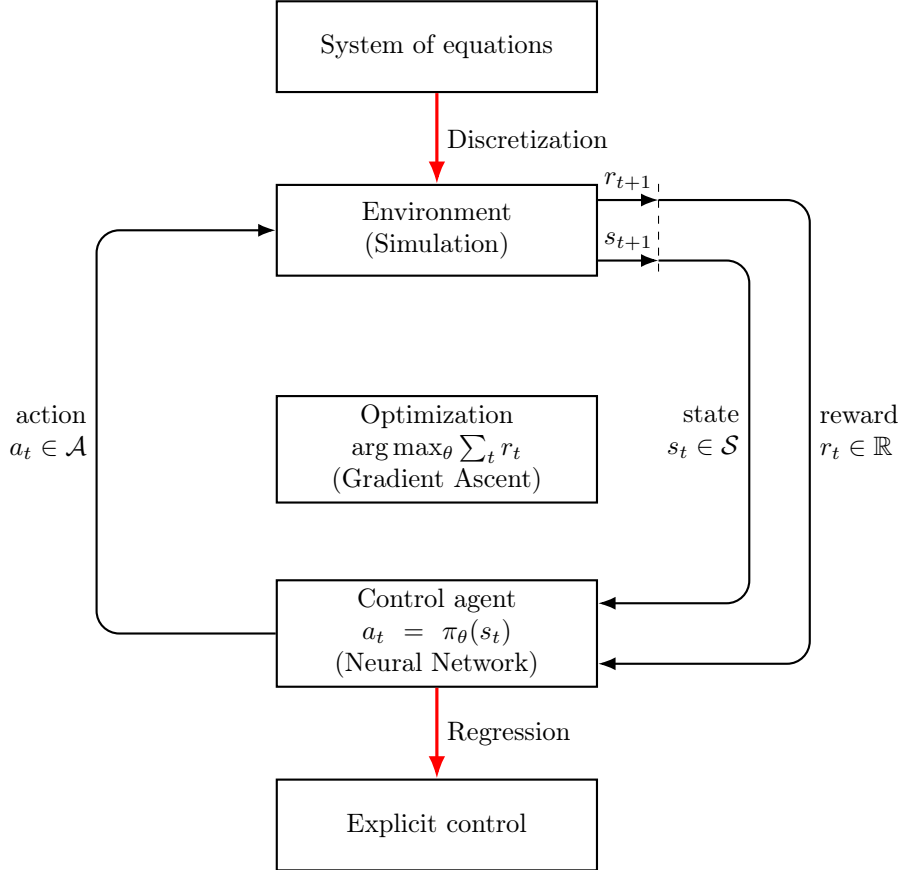


Figure 1.6: Diagram representing the (simplified) procedure by which we simulate our model in an environment that is used to train an RL agent, whose policy we then convert into an explicit control. The policy π_θ is modeled by a neural network with parameters θ , which takes a state as an input and outputs an action (or a distribution over actions in the stochastic case). The neural network is then optimized to maximize the sum of rewards it obtains over simulations.

1.2.2 Reaction-diffusion model in a bounded domain

In Chapter 6, we consider the spatial distribution of a mosquito population and study the release of sterile male mosquitoes. First, we examine the non-spatialized dynamics in the presence of an Allee effect (1.1), where the Allee term is given by $\frac{\eta M}{1+\eta M}$. For this model, we analyze the existence and stability of equilibria. This analysis is more complex due to the Allee term, which makes the equation for E dependent on M . We have identified the basic reproduction number R^1 given by

$$R^1 := \mathcal{R}_0 - \frac{2\delta_M}{\eta K(1-\nu)\nu_E} \left(1 + \sqrt{1 + \frac{\eta K(1-\nu)\nu_E}{\delta_M}} \right).$$

Unlike the basic reproduction number in other models, this one depends on additional parameters: K , η , and δ_M . We identified three equilibrium points depending on the values of R^1 . If $R^1 < 0$, the trivial equilibrium is the only equilibrium point, and we showed in Theorem 6.2.2 that it is globally asymptotically stable. When $R^1 > 1$, two non-trivial equilibria arise. The Routh-Hurwitz criterion allows us to conclude that one is locally stable while the other is unstable. Finally, when $R^1 = 1$, there exists a unique non-trivial equilibrium point $X_{E_0} = (E_0, F_0, M_0)$ that is globally stable on the set

$$\left\{ (E, F, M)^T \in [0, +\infty)^3 : E_0 \leq E \leq K, F_0 \leq F, \text{ and } M_0 \leq M \right\},$$

and unstable elsewhere. Following this analysis, we consider the spatialized model where Neumann boundary conditions are imposed on the states F , M , and M_s . It is worth noting that the choice to study the spatialized model with the Allee term $\frac{\eta M}{1+\eta M}$ is also motivated by the simplification of the mathematical analysis it offers, as it removes the singularity of the term $\frac{M}{M+\gamma M_s}$ present in the SIT model studied in the previous chapters. The dynamics of E are not affected by the diffusion operator since mosquito larvae do not diffuse during their aquatic phase. We study the existence and uniqueness of solutions for the two spatialized models (with or without sterile males, and with a feedback control law assumed to be Lipschitz continuous and at most linearly growing). The notion of solution used in this study is that of a weak solution for the states F , M , and M_s , defined via test functions of class C^1 . We use the fixed-point method along with certain functional estimates to solve this problem. Subsequently, we construct stabilizing feedback control laws, which we derived using the backstepping method.

Life cycle model of a mosquito population in a bounded domain

The motivation here is to prove the well-posedness of the system reaction-diffusion equations that models the spatial dispersion of a mosquito population in a bounded domain. We consider an established mosquito population; therefore, we do not apply the one-dimensional traveling wave simplification. Instead, we study the life cycle model of a mosquito population in a domain Ω , which is a non-empty, regular, open subset of $\mathbb{R}^2$. We select the space $L^1(\Omega)$ as our working space. In the context of mathematical biology, the space L^1 (the space of absolutely integrable functions) offers several advantages that make it particularly useful and often more relevant than L^2 and H^1 , which are typically chosen due to their convenience for mathematical analysis. We recall the definition of weak solution of this problem. Let $T > 0$. The application $(E, F, M)^T : [0, T] \times \Omega \rightarrow \mathbb{R}^3$ is a weak solution of the Cauchy problem (1.7) on $[0, T]$ if

$$E, F, \text{ and } M \text{ take values in } [0, +\infty), \quad (1.41)$$

$$E, F, \text{ and } M \text{ are in } C^0([0, T]; L^1(\Omega)), \quad (1.42)$$

$$F \text{ and } M \text{ are in } L^1((0, T); W^{1,1}(\Omega)), \quad (1.43)$$

and (6.29), (6.30) hold, and (6.26) and (6.27) hold for every $t \in [0, T]$ and for every $\varphi : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ of class C^1 . Moreover, $(E, F, M)^T : [0, +\infty) \times \Omega \rightarrow \mathbb{R}^3$ is a weak solution of the Cauchy problem (1.7) on $[0, +\infty)$ if, for every $T > 0$, its restriction to $[0, T] \times \Omega$ is a weak solution to (1.7) on $[0, T]$. The first challenge is the well-posedness of the Cauchy problem (1.7). The well-posedness result is that for $(E^0, F^0, M^0)^T : \Omega \rightarrow \mathbb{R}^3$ such that $E^0 \geq 0$, $F^0 \geq 0$, and $M^0 \geq 0$, $E^0 \in L^1(\Omega)$, $F^0 \in L^1(\Omega)$, and $M^0 \in L^1(\Omega)$, then there exists a weak solution of the Cauchy problem (1.7) on $[0, +\infty)$. Moreover, if

$$E^0 \in L^r(\Omega) \text{ for some } r \in (1, +\infty], F^0 \in L^1(\Omega), \text{ and } M^0 \in L^1(\Omega), \quad (1.44)$$

this weak solution is unique. Note that here the carrying capacity K is not necessarily constant. Instead, $K \in C(\Omega)$, and using the existence result of the Cauchy problem (1.7), one can study the mosquito life cycle under a heterogeneous carrying capacity, which is more realistic.

Reaction diffusion model for SIT

The main motivation here is to prove the well-posedness of the closed-loop control system and the asymptotic stability of the population near zero in the whole domain using a feedback law we have constructed. The weak solution notion is the same as defined for the life cycle model. We consider the feedback law (6.75). Our main results are: the closed-loop system (6.58), where u is given by (6.75), is non-negative (thanks to (6.76)) and is well-posed in $\mathcal{I}$ given by

$$\begin{aligned} \mathcal{I} := \left\{ (E, F, M, M_s)^T \in L^1(\Omega)^4 : E \geq 0, F \geq 0, M \geq 0, M_s \geq 0, E \in L^\infty(\Omega), \right. \\ \left. F \in L^r(\Omega) \text{ for some } r \in (1, +\infty), M \in L^1(\Omega), \text{ and } M_s \in L^1(\Omega) \right\}. \end{aligned} \quad (1.45)$$

Under the assumption that the diffusion coefficients are the same for wild and sterile males, the feedback law (6.75) globally and asymptotically stabilizes the control system to zero in the entire domain Ω .

This feedback law provides, at every position in the domain and at every time, the necessary density of sterile males to be released for stabilization. The numerical simulations show that even when the diffusion coefficients are different, this control law successfully stabilizes the population near zero.

1.3 Perspectives

The feedback control law we constructed using the backstepping method in Chapter 2 depends on two tuning parameters $\theta > 0$ and $\alpha > 0$. These parameters allow us to control the rate of convergence of the closed-loop system to zero. Knowing this decay rate can be useful when defining time horizons for stabilization and estimating the cost of releases. However, an optimal study of the release cost or application time as a function of these parameters was not addressed in our work. In our future research, we will explore this aspect.

Still within the framework of Chapter 2, the theoretical proof of global stability for the linear control laws $u = k(M + M_s)$ and $u = \lambda M$ also remains an open question.

In Chapter 3, we considered a more sophisticated SIT model whose states are: the density of the population in the aquatic phase E , the density of young females Y , the density of females fertilized by wild males F , the density of females fertilized by sterile males F_s , the density of wild males M , and the density of sterile males M_s . We built an observer system that requires only the measurements of M and M_s to estimate the other states, namely E , Y , F , and F_s . In our future work, we will consider the case where the measurement is $y = M + M_s$. In practice, this measurement is less time- and resource-consuming than the separate measurements of M and M_s . Other cases can also be studied, such as when the measurement is $y = M + M_s + F + F_s$. This measurement is also less costly than the previous ones. The main difficulty lies in the nonlinear term $\frac{M}{M + M_s}$.

In our study on the stabilization of the SIT model, another key objective is to construct feedback control laws that are easy to implement and inexpensive. Accordingly, in Chapter 4, we aimed to construct a control law that depends only on the measurement $y = M + M_s$. In our work, we theoretically tested the control law $u = k(M + M_s)$. Although, to date, no method in control theory allows one to determine a priori the state variables on which the control law should depend, this question has been explored through Reinforcement Learning. We will further investigate this question in future work, for example by considering the total number of adult mosquitoes $y = M + M_s + F + F_s$ as the observation. At the same time, we will continue our theoretical studies on this issue.

We have also undertaken the study of a spatialized SIT model. The goal is to go beyond simplified one-dimensional traveling wave models. However, the theoretical stability results we have obtained are based on the assumption that the diffusion coefficients of adult mosquitoes are the same. In practice, however, this assumption is questionable. Indeed, the diffusion of released sterile mosquitoes may be affected by the fact that they were reared in laboratory conditions from birth. Natural factors such as wind, rain, and other environmental conditions can also influence their movement within the target domain. Moreover, the release strategy will also significantly impact their spatial distribution. Therefore, to better reflect this reality, we intend to study, in our future work, the case where the diffusion coefficient is heterogeneous. As was done in the finite-dimensional SIT model case, we will also construct an observer system for the spatialized model in our future work.

Chapter 2

Global stabilization of a Sterile Insect Technique model by feedback laws

This chapter is a joint work with Luis Almeida and Jean-Michel Coron. It was published as an article in Journal of Optimization Theory and Applications [2].

Abstract. This work concerns feedback global stabilization of the sterile insect technique dynamics. The Sterile Insect Technique (SIT) is presently one of the most ecological methods for controlling insect pests responsible for crop destruction and disease transmission worldwide.

This technique consists in releasing sterile males among the insect pest population, the aim being to reduce fertility and, consequently, reduce significantly the wild insect population after a few generations.

In this work, we study the global stabilization of a pest population at extinction equilibrium by the SIT method and construct explicit feedback laws that stabilize the model. Numerical simulations show the efficiency of our feedback laws.

2.1 Introduction

Mosquitoes are known to transmit a variety of diseases such as malaria, dengue, yellow fever, Zika virus and many others. These diseases are responsible for a significant number of deaths worldwide. According to the World Health Organization (WHO), the number of malaria cases worldwide in 2022 was estimated at 249 million in 85 endemic countries and territories, an increase of 5 million compared to 2021. The estimated number of deaths in 2022 is 608,000 (see [85]). Dengue and Zika virus, also transmitted by mosquitoes, are estimated to cause hundreds of thousands of cases and thousands of deaths each year. In 2023, there was an unexpected increase in dengue cases, resulting in an all-time high of more than five million cases and more than 5,000 dengue-related deaths reported in more than 80 countries/territories and five WHO regions: Africa, the Americas, South-East Asia, the Western Pacific, and the Eastern Mediterranean (see [111]).

Although there are many effective vector control measures for malaria and arboviroses, some of them can have negative impact on the environment and may result in ecological damage. For example, insecticide spraying can have unintended effects on non-target organisms, including beneficial insects such as bees and butterflies [93, 64]. In addition, repeated use of insecticides often leads to the development of resistance in mosquito populations [86].

The sterile insect technique (SIT) has been proposed as an alternative tool for reducing mosquito populations. The technique involves sterilizing male mosquitoes (frequently this is done using ionizing radiation) and then releasing them into the wild to mate with wild females. This strategy was initially applied successfully (since the 1950s) to nearly eradicate the screw-worm fly in North America. Since then, this technique has also been used for different agricultural pests and disease vectors [23, 34, 110].

One advantage of using such a technique is that it only targets the desired species and also significantly reduces the impact on the ecosystem. This is why this technique is increasingly used for the control of insect pests and insect disease vectors. Some previous works have considered applications of feedback controls to SIT: impulsive feedback controls for a 3-D model [32, 33], optimal controls for a 2-D model [12] and even optimal impulsive controls for an epidemic model for a vector borne disease in the human population [10].

For the sake of simplicity, in this paper we chose to focus our presentation on the particular and important case of mosquito population control but many of the results presented can be extended to the use of SIT for the control of other pests.

In order to determine the appropriate releases of sterile males to approach the extinction equilibrium of the population, we use mathematical control theory which provides the necessary tools for constructing such a control. Our work involves building this feedback law starting from the model proposed in [102] without the Allee effect. Our theoretical results are illustrated with numerical simulations. Moreover, in section 2.4 we do a comparative study between the different feedback laws.

Remark 2.1.1. *While we were finishing writing this work, we learned that the reduced system (system of two ODE studied in [12]) was also recently studied by A. Cristofaro and L. Rossi in [46]. In particular, they were able to construct a feedback law leading to global stabilization of the extinction equilibrium in this setting using a backstepping approach.*

2.2 Mathematical modeling of mosquito population dynamics

2.2.1 Mathematical modeling of wild mosquito population dynamics

The life cycle of mosquitoes has many stages but we will consider a simplified model where we just separate an aquatic and an adult phase. The aquatic phase includes egg, larva and pupa stages. After the pupa stage, adult mosquitoes emerge and it is in the adult phase that mosquitoes reproduce and only female mosquitoes bite.

As a matter of fact, in order to lay their eggs, female mosquitoes need not only to be fertilized by males but also to have a blood meal. Thus, every 4-5 days, they will take a blood meal (that can sometimes involve biting several victims) and lay 100 to 150 eggs in different places (10 to 15 per place). An adult mosquito usually lives for 2 to 4 weeks. The mathematical model we present takes into account the two phases: the aquatic phase that we denote by E and the adult phase that we split into two sub-compartments, males, M , and females, F . We consider the dynamics presented in [102]. Based on this model and neglecting the Allee effect (i.e. taking $\beta = +\infty$ in system (2) of [102], which is the less favorable case for stabilizing the zero solution), we obtain the system

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E)E, \quad (2.1)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (2.2)$$

$$\dot{F} = \nu\nu_E E - \delta_F F, \quad (2.3)$$

where,

- $E(t) \geq 0$ is the mosquito density in aquatic phase at time t ;
- $M(t) \geq 0$ is the wild adult male density at time t ;
- $F(t) \geq 0$ is the density of adult females at time t ; we have supposed that all females are immediately fertilized in this setting and this equation is the only one that changes when we add the sterile males in which case only a fraction of the females will be fertilized;
- $\beta_E > 0$ is the oviposition rate;
- $\delta_E, \delta_M, \delta_F > 0$ are the death rates for eggs, wild adult males and fertilized females respectively;
- $\nu_E > 0$ is the hatching rate for eggs;
- $\nu \in (0, 1)$ the probability that a pupa gives rise to a female, and $(1 - \nu)$ is, therefore, the probability that it gives rise to a male. And, to simplify, we suppose females become fertilized immediately when they emerge from the pupal stage;
- $K > 0$ is the environmental capacity for eggs. It can be interpreted as the maximum density of eggs that females can lay in breeding sites. Since here the larval and pupal compartments are not present, it is as if E represents all the aquatic compartments, in which case in this term K represents a logistic law's carrying capacity for the aquatic phase that also includes the effects of competition between larvae. It has the dimensions of a spatial density.

We set $x = (E, M, F)^T$ and $\mathcal{D} = \mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x \geq 0\}$. The model (2.1)-(2.3) can be written in the form

$$\dot{x} = f(x), \quad (2.4)$$

where $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ represents the right hand side of (2.1)-(2.3). The map f is continuously differentiable on $\mathbb{R}^3$. Note that if $\dot{x} = f(x)$ and $x(0) \in \mathcal{D}$, then, for every $t \geq 0$, $x(t)$ is defined and belongs to $\mathcal{D}$. Setting the right

hand side of (2.1)-(2.3) to zero we obtain the extinction equilibrium $\mathbf{0} = (0, 0, 0)^T$ and the non-trivial equilibrium $x^* = (E^*, M^*, F^*)^T$ given by

$$E^* = K(1 - \frac{1}{\mathcal{R}_0}), \quad (2.5)$$

$$M^* = \frac{(1 - \nu)\nu_E}{\delta_M} E^*, \quad (2.6)$$

$$F^* = \frac{\nu\nu_E}{\delta_F} E^*, \quad (2.7)$$

where

$$\mathcal{R}_0 := \frac{\beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E)}. \quad (2.8)$$

Note that $x^* \in \mathcal{D}$ if and only if $\mathcal{R}_0 \geq 1$. Let us now recall some definitions connected to the stability of an equilibrium.

Definition 2.2.1. *Let $x_e \in \mathcal{D}$ be an equilibrium (of (2.4)). The equilibrium x_e is stable in $\mathcal{D}$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$(x_0 \in \mathcal{D} \text{ and } \|x(0) - x_e\| < \delta) \implies (\|x(t) - x_e\| < \varepsilon \text{ for all } t > 0). \quad (2.9)$$

The equilibrium x_e is unstable in $\mathcal{D}$ if it is not stable in $\mathcal{D}$. It is an attractor in $\mathcal{D}$ if there exists $\eta > 0$ such that, for every initial data $x(0)$ in $\mathcal{D}$ satisfying $\|x(0) - x_e\| < \eta$, $x(t) \rightarrow x_e$ as $t \rightarrow \infty$. It is a global attractor in $\mathcal{D}$ if, for every initial data in $\mathcal{D}$, $x(t) \rightarrow x_e$ as $t \rightarrow \infty$. It is locally asymptotically stable in $\mathcal{D}$ if it is both stable and an attractor in $\mathcal{D}$. Finally, it is globally asymptotically stable in $\mathcal{D}$ if it is both stable and a global attractor in $\mathcal{D}$.

The Jacobian of system (2.1)-(2.3) computed at the extinction equilibrium is

$$J(\mathbf{0}) = \begin{pmatrix} -(\nu_E + \delta_E) & 0 & \beta_E \\ (1 - \nu)\nu_E & -\delta_M & 0 \\ \nu\nu_E & 0 & -\delta_F \end{pmatrix}. \quad (2.10)$$

Its characteristic polynomial is

$$P(\lambda) = \lambda^3 + (\nu_E + \delta_E + \delta_M + \delta_F)\lambda^2 + ((\nu_E + \delta_E)\delta_F - \beta_E \nu \nu_E + \delta_M(\nu_E + \delta_E))\lambda + \delta_M((\nu_E + \delta_E)\delta_F - \beta_E \nu \nu_E). \quad (2.11)$$

Its roots are $-\delta_M$ and the roots of equation

$$\lambda^2 + (\nu_E + \delta_E + \delta_F)\lambda + \delta_F(\nu_E + \delta_E)(1 - \mathcal{R}_0) = 0 \quad (2.12)$$

If $\mathcal{R}_0 < 1$, all eigenvalues of $J(\mathbf{0})$ are either negative or have negative real parts, which implies that $\mathbf{0}$ is locally asymptotically stable. If $\mathcal{R}_0 = 1$ the eigenvalues of $J(\mathbf{0})$ are $-\delta_M$, 0, and $-(\nu_E + \delta_E + \delta_F) < 0$. If $\mathcal{R}_0 > 1$, the eigenvalues of $J(\mathbf{0})$ are all real, one is strictly positive, two are strictly negative.

The global stability properties of the extinction equilibrium $\mathbf{0} = (0, 0, 0)^T$ are described in terms of the basic offspring number $\mathcal{R}_0$ of the population. This is a key parameter in the theory of population dynamics. Depending on its value, more precisely if and only if $\mathcal{R}_0 > 1$, there exists a non-trivial equilibrium point (see [114, 12]). The essential properties of the model (2.1)-(2.3) are summarized in the following theorem similar to [18, Theorem 7] and [19, Theorem 1].

Theorem 2.2.1. *The following properties hold.*

- (P.1) *If $\mathcal{R}_0 \leq 1$, then $\mathbf{0} \in \mathbb{R}^3$ is a globally asymptotically stable equilibrium in $\mathcal{D}$ for (2.4);*
- (P.2) *If $\mathcal{R}_0 > 1$, then the system has two equilibria $\mathbf{0}$ and x^* in $\mathcal{D}$, where x^* is stable with basin of attraction $\mathcal{D} \setminus \{x = (E, M, F)^T \in \mathbb{R}_+^3 : E = F = 0\}$ and $\mathbf{0}$ is unstable in $\mathcal{D}$ with the non negative M - axis being a stable manifold.*

Proof. Let us first prove (P.1). We could proceed as in the proof of [18, Theorem 7 (i)] or [19, 1] in Theorem 1] which are based on properties of monotone operators. We propose a different approach, now based on Lyapunov functions. Let $t \mapsto x(t) = (E(t), M(t), F(t))^T$ be a solution of (2.4) defined at time 0 and such that $(E(0), M(0), F(0))^T \in \mathcal{D}$. One has

$$M(t) = e^{-\delta_M t} M(0) + (1 - \nu)\nu_E \int_0^t e^{-\delta_M(t-s)} E(s) ds, \quad (2.13)$$

which implies that

$$M(t) \leq M(0) + \frac{(1-\nu)\nu_E}{\delta_M} \sup\{E(s); s \geq 0\}, \quad (2.14)$$

$$M(t) \leq M(0)e^{-\delta_M t} + \frac{(1-\nu)\nu_E}{\delta_M} e^{-\delta_M t/2} \max\{E(s); s \in [0, t/2]\} + \frac{(1-\nu)\nu_E}{\delta_M} \sup\{E(s); s \geq t/2\}. \quad (2.15)$$

Inequality (2.14) shows that $\mathbf{0} \in \mathbb{R}^3$ is a stable equilibrium in $\mathcal{D}$ for (2.4) if $\mathbf{0} \in \mathbb{R}^2$ is a stable equilibrium in $[0, +\infty)^2$ for the subsystem in $(E, F)^T \in [0, +\infty)^2$:

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \quad (2.16)$$

$$\dot{F} = \nu \nu_E E - \delta_F F. \quad (2.17)$$

Inequality (2.15) shows that $\mathbf{0} \in \mathbb{R}^3$ is a global attractor in $\mathcal{D}$ for (2.4) if $\mathbf{0} \in \mathbb{R}^2$ is a global attractor in $[0, +\infty)^2$ for the subsystem (2.16)-(2.17) in $(E, F)^T \in [0, +\infty)^2$.

Hence, in order to prove **(P.1)**, it suffices to check that $\mathbf{0} \in [0, +\infty)^2$ is globally asymptotically stable in $[0, +\infty)^2$ for the system (2.16)-(2.17). To prove this last statement, let us consider the Lyapunov function $V : [0, +\infty)^2 \rightarrow \mathbb{R}$, $y = (E, F)^T \mapsto V(y)$, defined by

$$V(y) := \delta_F E + \beta_E F. \quad (2.18)$$

Then,

$$V \text{ is of class } \mathcal{C}^1, \quad (2.19)$$

$$V(y) > V((0, 0)^T) = 0, \quad \forall y \in [0, +\infty)^2 \setminus \{(0, 0)^T\}, \quad (2.20)$$

$$V(y) \rightarrow +\infty \text{ when } \|y\| \rightarrow +\infty \text{ with } y \in [0, +\infty)^2. \quad (2.21)$$

The time-derivative of V along the trajectories of (2.16)-(2.17) is

$$\dot{V} = -(\delta_F (\nu_E + \delta_E) - \beta_E \nu \nu_E) E - \frac{\delta_F \beta_E}{K} EF. \quad (2.22)$$

Let us now assume that

$$\mathcal{R}_0 \leq 1. \quad (2.23)$$

From (2.22) and (2.23) one gets

$$\dot{V} \leq -\frac{\delta_F \beta_E}{K} EF \leq 0. \quad (2.24)$$

We are going to conclude by using the LaSalle invariance principle. Let us assume that we have a trajectory $t \in \mathbb{R} \mapsto y(t) = (E(t), F(t))^T \in [0, +\infty)^2$ of (2.16)-(2.17) such that

$$\dot{V}(y(t)) = 0 \quad \forall t \in \mathbb{R}. \quad (2.25)$$

Then, using (2.24),

$$E(t)F(t) = 0 \quad \forall t \in \mathbb{R}. \quad (2.26)$$

Let us assume that there exists $t_0 \in \mathbb{R}$ such that

$$E(t_0) \neq 0. \quad (2.27)$$

Then there exists $\varepsilon > 0$ such that

$$E(t) \neq 0 \quad \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon), \quad (2.28)$$

which, together with (2.26), implies that

$$F(t) = 0 \quad \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon). \quad (2.29)$$

Differentiating (2.29) with respect to time and using (2.17) we get

$$E(t) = 0 \quad \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon), \quad (2.30)$$

in contradiction with (2.28). Hence

$$E(t) = 0 \quad \forall t \in \mathbb{R}. \quad (2.31)$$

Differentiating (2.31) with respect to time and using (2.16) we get that

$$F(t) = 0 \quad \forall t \in \mathbb{R}. \quad (2.32)$$

With the LaSalle invariance principle, this concludes the proof of **(P.1)**.

Remark 2.2.1. *In the case where $\mathcal{R}_0 < 1$ a simple linear strict Lyapunov function for the full system (2.4) is given in Remark 2.2.2.*

Let us now prove **(P.2)**. We first note that one has the following lemma, whose proof is obvious and is omitted.

Lemma 2.2.1. *Let $t \mapsto x(t) = (E(t), M(t), F(t))^T$ be a solution of (2.4) defined at time 0 and such that $(E(0), M(0), F(0))^T \in \mathcal{D}$. Then it is defined on $[0, +\infty)$. Moreover, if $E(0) \geq K$, then there exists one and only one time $t_0 \geq 0$ such that $E(t_0) = K$ and one has*

$$E(t) < K \quad \forall t > t_0. \quad (2.33)$$

Thanks to this lemma we are allowed to assume that $E < K$, which we do from now on. We then follow the proof of [18, Theorem 7 (ii)]. To prove the stability and basin of attraction of the non-trivial equilibrium x^* we use [98, Theorem 2.2 in Chapter 2]. This theorem applies to strongly monotone systems. The Jacobian (2.10) associated with (2.4) is not irreducible. Let us consider the subsystem for E and F , that is (2.16)-(2.17), which defines a dynamical system on $\mathbb{R}_+^2$. Its Jacobian

$$j((E, F)^T) = \begin{pmatrix} -(\nu_E + \delta_E) - \frac{\beta_E F}{K} & \beta_E(1 - \frac{E}{K}) \\ \nu \nu_E & -\delta_F \end{pmatrix} \quad (2.34)$$

is irreducible. Considering the usual coordinate-wise comparison and applying [98, Theorem 2.2 in Chapter 2] to the two dimensional interval

$$\{(E, F)^T \in \mathbb{R}_+^2 : 0 \leq E \leq E^*, 0 \leq F \leq F^*\}, \quad (2.35)$$

it follows that every solution starting in this interval, excluding the end points $(0, 0)^T$ and $(E^*, F^*)^T$, converges to one of the end points.

The Jacobian at $\mathbf{0} = (0, 0)^T$ is

$$j(\mathbf{0}) = \begin{pmatrix} -(\nu_E + \delta_E) & \beta_E \\ \nu \nu_E & -\delta_F \end{pmatrix} \quad (2.36)$$

Its characteristic equation is

$$\lambda^2 + (\delta_F + \nu_E + \delta_E)\lambda + \delta_F(\nu_E + \delta_E) - \beta_E \nu \nu_E = 0, \quad (2.37)$$

whose discriminant is

$$\Delta = (\nu_E + \delta_E - \delta_F)^2 + 4\beta_E \nu \nu_E \geq 0. \quad (2.38)$$

The eigenvalues are

$$\lambda_- := -\frac{(\delta_F + \nu_E + \delta_E) + \sqrt{\Delta}}{2} \quad (2.39)$$

$$\lambda_+ := -\frac{(\delta_F + \nu_E + \delta_E) - \sqrt{\Delta}}{2} \quad (2.40)$$

Therefore, since $\mathcal{R}_0 > 1$, $\lambda_+ > 0$ and so $\mathbf{0}$ is unstable. Since $j(\mathbf{0})$ is a Metzler matrix, it has a strictly positive eigenvector corresponding to the positive eigenvalue $\lambda_+ > 0$, which is

$$v_+ = \begin{pmatrix} 1 \\ \frac{(\nu_E + \delta_E - \delta_F) + \sqrt{\Delta}}{2\beta_E} \end{pmatrix} \quad (2.41)$$

Moreover, the eigenvector corresponding to the negative eigenvalue λ_- is

$$v_- = \begin{pmatrix} 1 \\ \frac{(\nu_E + \delta_E - \delta_F) - \sqrt{\Delta}}{2\beta_E} \end{pmatrix} \quad (2.42)$$

which has two components with opposite signs and is thus biologically meaningless. Hence, no solution converges to $\mathbf{0} = (0, 0)^T$ except the trivial solution which is identically equal to $\mathbf{0} = (0, 0)^T$. Therefore, every nontrivial solution converges to $(E^*, F^*)^T$. The implication for the three dimensional system (2.1)-(2.3) is that all solutions starting in the interval $[0, x^*]$, excluding the M -axis, converge to $x^* = (E^*, M^*, F^*)^T$.

Using the same argument as in [19], any solution starting at a point larger than x^* converges to x^* . Since any point in $\mathcal{D} \setminus \{x = (E, M, F)^T \in \mathbb{R}_+^3 : E = F = 0\}$ can be placed between a point below x^* , but not on the M -axis, and a point above x^* , every solution starting in $\mathcal{D} \setminus \{x = (E, M, F)^T \in \mathbb{R}_+^3 : E = F = 0\}$ converges to x^* . The monotone convergence of the solutions initiated below and above x^* implies the stability of x^* as well. This concludes the proof of **(P.2)** and of Theorem 2.2.1. $\square$

2.2.2 SIT model in mosquito population dynamics

The SIT model obtained neglecting the Allee effect from the one presented in [102] is

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E)E, \quad (2.43)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (2.44)$$

$$\dot{F} = \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F, \quad (2.45)$$

$$\dot{M}_s = u - \delta_s M_s, \quad (2.46)$$

where $M_s(t) \geq 0$ is the density of sterilized adult males, $\delta_s > 0$ is the death rate of sterilized adults, $u \geq 0$ is the control which is the density of sterile males released at time t , and $0 < \gamma_s \leq 1$ accounts for the fact that females may have a preference for fertile males. Then, the probability that a female mates with a fertile male is $M/(M + \gamma_s M_s)$. From now on we assume that

$$\delta_s \geq \delta_M, \quad (2.47)$$

which is a biologically relevant assumption (and even if this were not so, the sterile males would have a competitive advantage due to a higher longevity that would make SIT more efficient).

Let $\mathcal{D}' := [0, +\infty)^4$. When applying a feedback law $u : \mathcal{D}' \rightarrow [0, +\infty)$, the closed-loop system is the system

$$\dot{x} = H(x, u(x)), \quad (2.48)$$

where

$$H(x, u) = \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \\ u - \delta_s M_s \end{pmatrix}. \quad (2.49)$$

Concerning the regularity of the feedback law, we always assume that

$$u \in L_{\text{loc}}^\infty(\mathcal{D}'). \quad (2.50)$$

Note that, even if u is of class $\mathcal{C}^\infty$, the map $x \in \mathcal{D}' \mapsto H(x, u(x)) \in \mathbb{R}^4$ is not continuous and one needs to specify the definition of the solutions for the closed-loop system (2.48). Carathéodory solutions seem to be natural candidates. Roughly speaking, Carathéodory solutions are absolutely continuous curves that satisfy the integral version of the differential equation. These solutions are indeed useful in other contexts. However, if they can lead to robustness for small errors on the control, as shown in [16], they may not be robust with respect to arbitrary small measurement errors on the state, which is crucial for the application. To have a robustness with respect to arbitrary small measurement errors on the state, as shown in [60] (see also [45]), the good definition of the solutions for the closed-loop system (2.48) are the Filippov solutions, i.e. the solution of

$$\dot{x} \in Y(x) := \bigcap_{\varepsilon > 0} \bigcap_{N \in \mathcal{N}} \overline{\text{conv}} [X((x + \varepsilon B) \cap \mathcal{D}') \setminus N], \quad (2.51)$$

where

- B is the unit ball of $\mathbb{R}^4$;

- for a set A , $\overline{\text{conv}}[A]$ is the smaller closed convex set containing A ;
- $\mathcal{N}$ is the set of subsets of $\mathbb{R}^4$ of zero Lebesgue measure.
- $X(x) := H(x, u(x))$.

Let us recall that $x : I \subset \mathbb{R} \rightarrow \mathbb{R}^4$, $t \in I \mapsto x(t) \in \mathbb{R}^4$ (where I is an interval of $\mathbb{R}$) is a solution of (2.51) if $x \in W_{\text{loc}}^{1,\infty}(I)$ and is such that

$$\dot{x}(t) \in Y(x(t)) \text{ for almost every } t \in I. \quad (2.52)$$

For references about Filippov solutions, let us mention, in particular, [51, 52] and [22, Chapter 1]. For the definition of stability, global attractor and asymptotic stability, we use again Definition 2.2.1 (with $\mathcal{D}'$ instead of $\mathcal{D}$) and take now into account all the solutions in the Filippov sense in this definition. The motivation for using Filippov solutions is given in [45, Proposition 1.4]. The global asymptotic stability in this Filippov sense implies the existence of a Lyapunov function [41]; see also [45, Lemma 2.2]. This automatically gives some robustness properties with respect to (small) perturbations (including small measurement errors on the state), which is precisely the goal of feedback laws. In fact, for many feedback laws constructed in this article, an explicit Lyapunov function will be given, which allows to quantify this robustness.

Let us emphasize that in our case the Filippov solutions of our closed-loop system have the following properties

$$((E(0), F(0)) = (0, 0)) \implies ((E(t), F(t)) = (0, 0) \forall t \geq 0), \quad (2.53)$$

$$((E(0), F(0)) \neq (0, 0)) \implies (E(t) > 0, M(t) > 0, F(t) > 0 \forall t > 0). \quad (2.54)$$

From now on, the solutions of the closed-loop systems considered in this article are always the Filippov solutions.

Proposition 2.2.1 (See[12]: Stability properties of the system (2.43)-(2.46)). *Let us assume that*

$$\mathcal{R}_0 > 1. \quad (2.55)$$

Then the following properties hold.

1. *If $u = 0$, we have two equilibria:*

- *the extinction equilibrium $\mathbf{0}$, where $E = F = M = M_s = 0$, which is linearly unstable;*
- *the persistence equilibrium*

$$E^* = K(1 - \frac{1}{\mathcal{R}_0}), \quad (2.56)$$

$$M^* = \frac{(1 - \nu)\nu_E}{\delta_M} E^*, \quad (2.57)$$

$$F^* = \frac{\nu\nu_E}{\delta_F} E^*, \quad (2.58)$$

$$M_s^* = 0, \quad (2.59)$$

which is locally asymptotically stable.

2. *If $u \geq 0$, then the corresponding solution (E, M, F, M_s) to System (2.43)-(2.46) enjoys the following stability property:*

$$\begin{cases} E(0) \in (0, E^*], \\ M(0) \in (0, M^*], \\ F(0) \in (0, F^*], \\ M_s(0) \geq 0, \end{cases} \implies \begin{cases} E(t) \in (0, E^*], \\ M(t) \in (0, M^*], \\ F(t) \in (0, F^*], \\ M_s(t) \geq 0, \end{cases} \text{ for all } t \geq 0. \quad (2.60)$$

Let

$$U^* = \mathcal{R}_0 \frac{K(1 - \nu)\nu_E \delta_s}{4\gamma_s \delta_M} (1 - \frac{1}{\mathcal{R}_0})^2. \quad (2.61)$$

If $u(\cdot)$ denotes a constant control function equal to some $\bar{U} > U^*$ for all $t \geq 0$, then the corresponding solution $(E(t), M(t), F(t), M_s(t))$ converges to $(0, 0, 0, \bar{U}/\delta_s)$ as $t \rightarrow \infty$.

Concerning the global asymptotic stability of $\mathbf{0}$ for the system (2.43)-(2.46) in $\mathcal{D}' := [0, +\infty)^4$, using a Lyapunov approach, one can get the following theorem.

Theorem 2.2.2. *Let $u = 0$. If $\mathcal{R}_0 < 1$, then $\mathbf{0}$ is globally asymptotically stable in $\mathcal{D}'$ for the system (2.43)-(2.46).*

Proof. Let $x = (E, M, F, M_s)^T$. We are going to conclude by applying Lyapunov's second theorem. To do so, a candidate Lyapunov function is $V : \mathcal{D}' \rightarrow \mathbb{R}_+$, $x \mapsto V(x)$, defined by

$$V(x) := \frac{1 + \mathcal{R}_0}{1 - \mathcal{R}_0} E + \frac{2\beta_E}{\delta_F(1 - \mathcal{R}_0)} F + M + M_s. \quad (2.62)$$

Note that, since $\mathcal{R}_0 < 1$,

$$V(x) > V(\mathbf{0}) = 0, \quad \forall x \in \mathcal{D}' \setminus \{\mathbf{0}\}, \quad (2.63)$$

$$V(x) \rightarrow +\infty \text{ as } |x| \rightarrow +\infty \text{ with } x \in \mathcal{D}'. \quad (2.64)$$

Moreover, along the trajectories of (2.43)-(2.46),

$$\begin{aligned} \dot{V}(x) = & -(\nu\nu_E + \delta_E)E - \frac{\beta_E}{K} \frac{1 + \mathcal{R}_0}{1 - \mathcal{R}_0} FE - \delta_M M - \beta_E F - \delta_s M_s \\ & - \frac{2\beta_E \nu\nu_E}{\delta_F(1 - \mathcal{R}_0)} \frac{\gamma_s M_s}{M + \gamma_s M_s} E, \text{ if } M + M_s \neq 0. \end{aligned} \quad (2.65)$$

From (2.62) and (2.65), one gets

$$\dot{V}(x) \leq -c_0 V(x) \text{ if } M + M_s \neq 0, \quad (2.66)$$

with

$$c_0 := \min \left\{ \frac{(\nu\nu_E + \delta_E)(1 - \mathcal{R}_0)}{1 + \mathcal{R}_0}, \frac{\delta_F(1 - \mathcal{R}_0)}{2}, \delta_M, \delta_s \right\} \quad (2.67)$$

Let us point out that, for every solution $t \mapsto x(t) = (E(t), M(t), F(t), M_s(t))^T$ of the closed-loop system (2.43)-(2.46) defined at time 0 and such that $x(0) \in \mathcal{D}'$,

$$(M(0) + M_s(0) > 0) \implies (M(t) + M_s(t) > 0, \quad \forall t > 0), \quad (2.68)$$

$$(x(0) = 0) \implies (x(t) = 0, \quad \forall t \geq 0). \quad (2.69)$$

From (2.53), (2.54), (2.63), (2.66), (2.68) and (2.69), one has, for every solution $t \mapsto x(t) = (E(t), M(t), F(t), M_s(t))^T$ of the closed-loop system (2.43)-(2.46) defined at time 0 and such that $x(0) \in \mathcal{D}'$,

$$V(x(t)) \leq V(x(0))e^{-c_0 t} \quad \forall t \geq 0, \quad (2.70)$$

which, together with (2.63) and (2.64), concludes the proof of Theorem 2.2.2 (and even shows the global exponential stability and provides an estimate on the exponential decay rate c_0 given by (2.67)). $\square$

Remark 2.2.2. Note that Theorem 2.2.2 implies Theorem 2.2.1 in the case $\mathcal{R}_0 < 1$ and our proof of Theorem 2.2.2 provides, for this case, a (strict) Lyapunov function which is just

$$\tilde{V}((E, M, F)^T) := \frac{1 + \mathcal{R}_0}{1 - \mathcal{R}_0} E + \frac{2\beta_E}{\delta_F(1 - \mathcal{R}_0)} F + M. \quad (2.71)$$

It would be interesting to provide Lyapunov functions for the two remaining cases $\mathcal{R}_0 = 1$ and $\mathcal{R}_0 > 1$.

2.3 Global stabilization by feedback of the extinction equilibrium

2.3.1 Backstepping feedback

For the backstepping method, the control system has the following structure:

$$\dot{x}_1 = f(x_1, x_2), \quad (2.72)$$

$$\dot{x}_2 = u - g(x_1, x_2), \quad (2.73)$$

where the state is $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^m$ and the control is $u \in \mathbb{R}^m$. The key and classical theorem for backstepping is the following one (see, for instance, [21, Theorem 19.2, page 110] or [44, Theorem 12.24, page 334]).

Theorem 2.3.1. Assume that f and g are of class $\mathcal{C}^1$ and that for the control system

$$\dot{x}_1 = f(x_1, v), \quad (2.74)$$

where the state is $x_1 \in \mathbb{R}^p$ and the control is $v \in \mathbb{R}^m$, $\mathbf{0} \in \mathbb{R}^p$ can be globally asymptotically stabilized by means of a feedback law $x_1 \in \mathbb{R}^p \mapsto v(x_1) \in \mathbb{R}^m$ of class $\mathcal{C}^1$. Then, for the control system (2.72)-(2.73), $\mathbf{0} \in \mathbb{R}^p \times \mathbb{R}^m$ can be globally asymptotically stabilized by means of a continuous feedback law $x \in \mathbb{R}^p \times \mathbb{R}^m \mapsto u(x) \in \mathbb{R}^m$.

Let $x := (E, M, F)^T$. One way to rewrite the dynamics (2.43)-(2.46) is

$$\begin{cases} \dot{x} = f(x, M_s), \\ \dot{M}_s = u - \delta_s M_s, \end{cases} \quad (2.75)$$

where

$$f(x, M_s) := \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E \\ (1 - \nu) \nu_E E - \delta_M M \\ \nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \end{pmatrix}. \quad (2.76)$$

As f is not of class $\mathcal{C}^1(\mathcal{D} \times [0, +\infty))$ and the feedback law has to be non-negative, we cannot directly apply the backstepping theorem. However, to build the feedback law we use the classical Lyapunov approach of the proof of Theorem 2.3.1 (see, for example, [44, pages 334–335]) allowing us to select an appropriate control. Unfortunately, the control that we get with this approach is not positive all the time. To get around this, using the same Lyapunov function, we propose a new feedback law that is non-negative, decreases the Lyapunov function and leads to global asymptotic stability of the extinction equilibrium.

First, consider the control system $\dot{x} = f(x, M_s)$ with the state being $x \in \mathcal{D}$ and the control being $M_s \in [0, +\infty)$. We assume that M_s is of the form $M_s = \theta M$ and study the closed-loop system

$$\dot{x} = f(x, \theta M). \quad (2.77)$$

We have

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \\ \dot{M} = (1 - \nu) \nu_E E - \delta_M M, \\ \dot{F} = \frac{\nu \nu_E}{1 + \gamma_s \theta} E - \delta_F F. \end{cases} \quad (2.78)$$

It is a smooth dynamical system on $\mathcal{D} = [0, +\infty)^3$ which is also a positively invariant set for this dynamical system.

Setting the right hand side of (2.78) to zero we obtain the equilibrium $\mathbf{0} \in [0, +\infty)^3$ and the non-trivial equilibrium $x^{**} = (E^{**}, M^{**}, F^{**})$ given by

$$E^{**} = K \left(1 - \frac{1}{\mathcal{R}(\theta)}\right), \quad (2.79)$$

$$M^{**} = \frac{(1 - \nu) \nu_E}{\delta_M} E^{**}, \quad (2.80)$$

$$F^{**} = \frac{\nu \nu_E}{\delta_F (1 + \gamma_s \theta)} E^{**}, \quad (2.81)$$

where the offspring number is now

$$\mathcal{R}(\theta) := \frac{\beta_E \nu \nu_E}{\delta_F (1 + \gamma_s \theta) (\nu_E + \delta_E)} = \frac{\mathcal{R}_0}{1 + \gamma_s \theta}. \quad (2.82)$$

Note that if $\mathcal{R}(\theta) \leq 1$, $\mathbf{0} \in \mathbb{R}^3$ is the only equilibrium point of the system in $\mathcal{D}$.

Our next proposition shows that the feedback law $M_s = \theta M$ stabilizes our control system $\dot{x} = f((x^T, M_s)^T)$ if $\mathcal{R}(\theta) < 1$.

Proposition 2.3.1. *Assume that*

$$\mathcal{R}(\theta) < 1. \quad (2.83)$$

Then $\mathbf{0}$ is globally asymptotically stable in $\mathcal{D}$ for system (2.77).

Proof. We apply Lyapunov's second theorem. To do so, we define

$$\begin{aligned} V : x \in [0, +\infty)^3 &\mapsto V(x) \in \mathbb{R}_+, \\ V(x) &:= \frac{1 + \mathcal{R}(\theta)}{1 - \mathcal{R}(\theta)} E + M + \frac{2\beta_E}{\delta_F (1 - \mathcal{R}(\theta))} F. \end{aligned} \quad (2.84)$$

As (2.83) holds,

$$V \text{ is of class } \mathcal{C}^1, \quad (2.85)$$

$$V(x) > V((0, 0, 0)^T) = 0, \quad \forall x \in [0, +\infty)^3 \setminus \{(0, 0, 0)^T\}, \quad (2.86)$$

$$V(x) \rightarrow +\infty \text{ when } \|x\| \rightarrow +\infty \text{ with } x \in \mathcal{D}. \quad (2.87)$$

We have

$$\dot{V}(x) = \nabla V(x) \cdot f(x, \theta M) = \begin{pmatrix} \frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)} \\ 1 \\ \frac{2\beta_E}{\delta_F(1-\mathcal{R}(\theta))} \end{pmatrix}^T \cdot \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) - aE \\ cE - \delta_M M \\ \frac{\nu\nu_E}{1+\gamma_s\theta} E - \delta_F F \end{pmatrix}. \quad (2.88)$$

So

$$\dot{V}(x) = -\beta_E F - \delta_M M - \frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)} \frac{\beta_E}{K} F E - (\nu\nu_E + \delta_E) E. \quad (2.89)$$

Then, using once more (2.83), we get the existence of $c > 0$ such that

$$\dot{V}(x) \leq -cV(x), \quad \forall x \in [0, +\infty)^3. \quad (2.90)$$

This concludes the proof of Proposition 2.3.1. $\square$

Let us define

$$\psi := \frac{2\beta_E \nu \nu_E}{\delta_F(1-\mathcal{R}(\theta))(1+\gamma_s\theta)}, \quad (2.91)$$

and, for α and β_s (the latter having dimension of a rate) chosen in $(0, +\infty)$, the map $G : \mathcal{D}' := [0, +\infty)^4 \rightarrow \mathbb{R}$, $(x^T, M_s)^T \mapsto G((x^T, M_s)^T)$ by

$$\begin{aligned} G((x^T, M_s)^T) &:= \frac{\gamma_s \psi E(\theta M + M_s)^2}{\alpha(M + \gamma_s M_s)(3\theta M + M_s)} \\ &\quad + \frac{((1-\nu)\nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s)}{3\theta M + M_s} \\ &\quad + \delta_s M_s + \frac{\beta_s}{\alpha}(\theta M - M_s), \text{ if } M + M_s \neq 0, \end{aligned} \quad (2.92)$$

$$G((x^T, M_s)^T) := 0, \text{ if } M + M_s = 0. \quad (2.93)$$

Finally, let us define the feedback law $u : \mathcal{D}' \rightarrow [0, +\infty)$, $(x^T, M_s)^T \mapsto u((x^T, M_s)^T)$, by

$$u((x^T, M_s)^T) := \max\left(0, G((x^T, M_s)^T)\right). \quad (2.94)$$

Note that u , which is Lebesgue measurable, is not continuous in $\mathcal{D}'$. However

$$\text{there exists } C > 0 \text{ such that } |u(y)| \leq C\|y\| \quad \forall y \in \mathcal{D}'. \quad (2.95)$$

Property (2.95) is important for the applications since it implies that the density u of sterile males released is going to be small when the state is close to $\mathbf{0}$. For instance, this is essential to reduce the number of mosquitoes necessary for a long term intervention and also to allow using the sterile mosquitoes which are no longer needed in an area where the population is already close to zero, to intervene in other zones.

This is in contrast with the constant control in Proposition 2.2.1. Property (2.95) also implies that $u \in L_{\text{loc}}^\infty(\mathcal{D}')$, which allows to consider Filippov solutions for the closed-loop system, i.e. the system (2.43)-(2.46) with the feedback law (2.94).

The next theorem shows that the feedback law (2.94) stabilizes the control system (2.43)-(2.46).

Theorem 2.3.2. *Assume that (2.83) holds. Then $\mathbf{0} \in \mathcal{D}'$ is globally asymptotically stable in $\mathcal{D}'$ for system (2.43)-(2.46) with the feedback law (2.94).*

Proof. Let us define $W : \mathcal{D}' \rightarrow \mathbb{R}$ by

$$W((x^T, M_s)^T) := V(x) + \alpha \frac{(\theta M - M_s)^2}{\theta M + M_s}, \text{ if } M + M_s \neq 0, \quad (2.96)$$

$$W((x^T, M_s)^T) := V(x), \text{ if } M + M_s = 0. \quad (2.97)$$

We have

$$W \text{ is continuous,} \quad (2.98)$$

$$W \text{ is of class } \mathcal{C}^1 \text{ on } \mathcal{D}' \setminus \left\{ (E, M, F, M_s)^T \in \mathcal{D}'; M + M_s = 0 \right\}, \quad (2.99)$$

$$\begin{aligned} W((x^T, M_s)^T) &\rightarrow +\infty, \text{ as } \|x\| + M_s \rightarrow +\infty, \\ &\text{with } x \in \mathcal{D} \text{ and } M_s \in [0, +\infty), \end{aligned} \quad (2.100)$$

$$W((x^T, M_s)^T) > W(\mathbf{0}) = 0, \forall (x^T, M_s)^T \in \mathcal{D}' \setminus \{\mathbf{0}\}. \quad (2.101)$$

From now on, and until the end of this proof, we assume that $(x^T, M_s)^T$ is in $\mathcal{D}'$ and until (2.112) below we further assume that

$$(M, M_s) \neq (0, 0). \quad (2.102)$$

One has

$$\begin{aligned} \dot{W}((x^T, M_s)^T) &= \nabla V(x) \cdot f(x, M_s) + \alpha(\theta M - M_s) \\ &\quad \frac{2(\theta \dot{M} - \dot{M}_s)(\theta M + M_s) - (\theta \dot{M} + \dot{M}_s)(\theta M - M_s)}{(\theta M + M_s)^2} \\ &= \nabla V(x) \cdot f(x, \theta M) + \nabla V(x) \cdot (f(x, M_s) - f(x, \theta M)) \\ &\quad + \alpha(\theta M - M_s) \frac{\theta \dot{M}(\theta M + 3M_s) - \dot{M}_s(3\theta M + M_s)}{(\theta M + M_s)^2}. \end{aligned}$$

$$\nabla V(x) \cdot (f(x, M_s) - f(x, \theta M)) =$$

$$\begin{aligned} &\left(\begin{array}{c} \frac{1 + \mathcal{R}(\theta)}{1 - \mathcal{R}(\theta)} \\ 1 \\ \frac{2\beta_E}{\delta_F(1 - \mathcal{R}(\theta))} \end{array} \right)^T \cdot \left(\begin{array}{c} 0 \\ 0 \\ \frac{\nu \nu_E \gamma_s E(\theta M - M_s)}{(M + \gamma_s M_s)(1 + \gamma_s \theta)} \end{array} \right) = \\ &\quad \frac{\psi \gamma_s E(\theta M - M_s)}{M + \gamma_s M_s}, \end{aligned} \quad (2.103)$$

$$\begin{aligned} \dot{W}((x^T, M_s)^T) &= \nabla V(x) \cdot f(x, \theta M) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \\ &\quad \left[\frac{(\nabla V(x) \cdot (f((x^T, M_s)^T) - f(x, \theta M)))(\theta M + M_s)^2}{\alpha(\theta M - M_s)} \right. \\ &\quad \left. + \theta \dot{M}(\theta M + 3M_s) - \dot{M}_s(3\theta M + M_s) \right] \end{aligned}$$

$$= \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\psi \gamma_s E(\theta M + M_s)^2}{\alpha(M + \gamma_s M_s)} \right. \quad (2.104)$$

$$\begin{aligned} &+ ((1 - \nu) \nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \\ &\left. - u(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s) \right]. \end{aligned} \quad (2.105)$$

We take u as given by (2.94).

Therefore, in case

$$\begin{aligned} &\frac{\psi \gamma_s E(\theta M + M_s)^2}{\alpha(M + \gamma_s M_s)} + ((1 - \nu) \nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \\ &\quad + \delta_s M_s(3\theta M + M_s) + \frac{\beta_s}{\alpha}(\theta M - M_s)(3\theta M + M_s) > 0, \end{aligned} \quad (2.106)$$

we have

$$\begin{aligned} u &= \frac{1}{3\theta M + M_s} \left[\frac{\psi \gamma_s E(\theta M + M_s)^2}{\alpha(M + \gamma_s M_s)} + ((1 - \nu) \nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \right. \\ &\quad \left. + \delta_s M_s(3\theta M + M_s) + \frac{\beta_s}{\alpha}(\theta M - M_s)(3\theta M + M_s) \right], \end{aligned}$$

which, together with (2.105), leads to

$$\dot{W}((x^T, M_s)^T) = \dot{V}(x) - \beta_s \frac{(\theta M - M_s)^2 (3\theta M + M_s)}{(\theta M + M_s)^2}. \quad (2.107)$$

Otherwise, i.e. if (2.106) does not hold,

$$\begin{aligned} & \frac{\psi \gamma_s E (\theta M + M_s)^2}{\alpha (M + \gamma_s M_s)} + ((1 - \nu) \nu_E \theta E - \theta \delta_M M) (\theta M + 3M_s) \\ & + \delta_s M_s (3\theta M + M_s) + \frac{\beta_s}{\alpha} (\theta M - M_s) (3\theta M + M_s) \leq 0, \end{aligned} \quad (2.108)$$

so, by (2.94),

$$u = 0. \quad (2.109)$$

We consider two cases:

Case 1: $\theta M > M_s$

Using (2.105), (2.108) and (2.109)

$$\dot{W}((x^T, M_s)^T) \leq \dot{V}(x) - \beta_s \frac{(\theta M - M_s)^2 (3\theta M + M_s)}{(\theta M + M_s)^2}. \quad (2.110)$$

Case 2: $\theta M \leq M_s$

Using once more (2.105) and (2.109)

$$\begin{aligned} \dot{W}((x^T, M_s)^T) = \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} & \left[\frac{\psi \gamma_s E (\theta M + M_s)^2}{\alpha (M + \gamma_s M_s)} \right. \\ & \left. + \theta ((1 - \nu) \nu_E E - \delta_M M) (\theta M + 3M_s) + \delta_s M_s (3\theta M + M_s) \right]. \end{aligned} \quad (2.111)$$

Using (2.47)

$$-\delta_M M (\theta M + 3M_s) + \delta_s M_s (3\theta M + M_s) \geq \delta_M (M_s - \theta M) (M_s + \theta M),$$

which, together with (2.111), implies that

$$\dot{W}((x^T, M_s)^T) \leq \dot{V}(x) - \alpha \delta_M \frac{(\theta M - M_s)^2}{(\theta M + M_s)}. \quad (2.112)$$

To summarize, using (2.90), (2.107), (2.110) and (2.112), one gets the existence of $c' > 0$, independent of $(x^T, M_s)^T \in \mathcal{D}'$, such that

$$\dot{W}((x^T, M_s)^T) \leq -c' W((x^T, M_s)^T) \text{ if } M + M_s \neq 0. \quad (2.113)$$

Since one still has (2.53), (2.54), (2.68) and (2.69) (for $x = (x^T, M_s^T)^T$), this proves Theorem 2.3.2 as in the proof of Theorem 2.2.2 (and, again, even gives the global exponential stability and provides an estimate on the exponential decay rate). $\square$

Remark 2.3.1. *It is important to note that the backstepping feedback control (2.94) does not depend on the environmental capacity K , which is can also be an interesting feature for the field applications.*

Numerical simulations

The numerical simulations of the dynamics when applying the feedback (2.94) are shown in figure 2.3.1. The parameters we use are set in table 2.1. The condition (2.83) gives $\theta > 75.5625$. We fix $K = 22200 \text{ ha}^{-1}$ and we consider the persistence equilibrium as initial condition. That gives $E^0 = 21910, M^0 = 5587, F^0 = 13419$ and $M_s^0 = 0$. We take $\theta = 220, \alpha = 13$ and $\beta_s = 1 \text{ Day}^{-1}$.

Parameter	Name	Value interval	Chosen value	Unity
β_E	Effective fecundity	7.46-14.85	10	Day ⁻¹
γ_s	Mating competitiveness of sterilized males	0-1	1	-
ν_E	Hatching parameter	0.005-0.25	0.05	Day ⁻¹
δ_E	Mosquitoes in aquatic phase death rate	0.023-0.046	0.03	Day ⁻¹
δ_F	Female death rate	0.033-0.046	0.04	Day ⁻¹
δ_M	Males death rate	0.077-0.139	0.1	Day ⁻¹
δ_s	Sterilized male death rate		0.12	Day ⁻¹
ν	Probability of emergence		0.49	

Table 2.1: Value intervals for the parameters of system (2.43)-(2.46) (see [102])

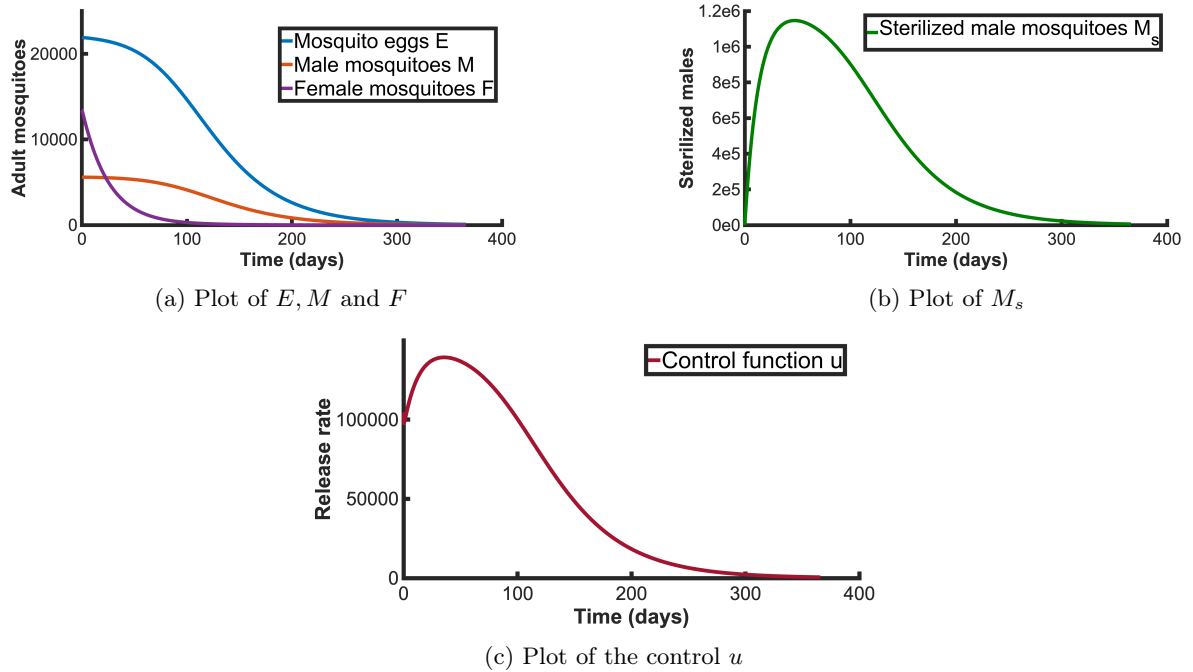


Figure 2.3.1. (a): Plot of E , M and F when applying the feedback (2.94), the initial condition being the persistence equilibrium. (b): Plot of M_s . (c): Plot of the feedback control function u .

In this case, with $t_f = 360$ days,

$$\int_0^{t_f} u(t) dt \approx 18 \text{ millions.} \quad (2.114)$$

Robustness test

To analyze the robustness of our feedback law we use the following protocol: the feedback law is given by (2.94) with fixed values of the parameters corresponding to the ones chosen in table 2.1, but for computing the real dynamics of the system (2.48) we consider simultaneous random perturbations of the system parameters with the

following distribution

$$\begin{aligned}
 \hat{\beta}_E &\sim \mathcal{U}(7.46, 14.85), \\
 \hat{\nu}_E &\sim \mathcal{U}(0.005, 0.25), \\
 \hat{\delta}_E &\sim \mathcal{U}(0.023, 0.046), \\
 \hat{\delta}_F &\sim \mathcal{U}(0.033, 0.046), \\
 \hat{\delta}_M &\sim \mathcal{U}(0.077, 0.139), \\
 \hat{\delta}_s &\sim \mathcal{U}(0.077, 0.139), \\
 \hat{\gamma}_s &\sim \mathcal{U}(0.5, 1.0),
 \end{aligned} \tag{2.115}$$

where $\mathcal{U}(a, b)$ is the uniform distribution on interval $[a, b]$.

Figure 2.3.2 shows 200 simulations with random initial conditions in $[0, 10K]^4$.

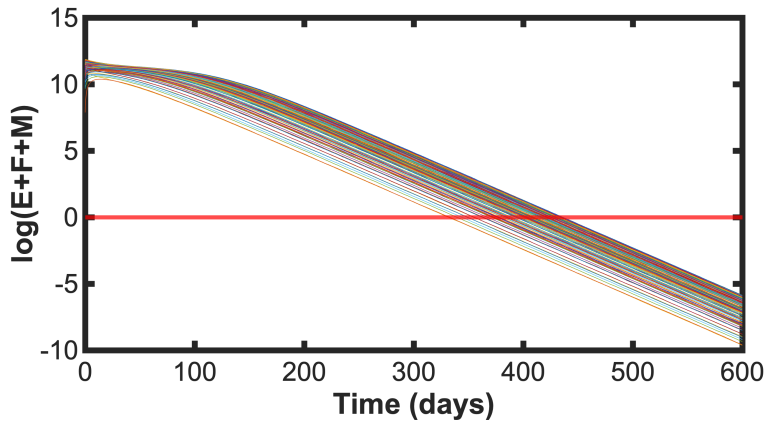


Figure 2.3.2. Evolution of $\log(E + F + M)$ for 200 random initial conditions when applying feedback (2.94) computed with $\theta = 220$, $\alpha = 13$, $\beta = 1 \text{ Day}^{-1}$ and the parameters given in table 2.1, which are not the parameters used for simulating the mosquito dynamics (the latter being taken randomly for each simulation according to (2.115))

We observe that the feedback (2.94) is robust: it still stabilizes the dynamics at extinction equilibrium if the changes in the parameters are not too large.

To apply the feedback (2.94) we must estimate the number of male and female mosquitoes and the number of eggs. Some techniques used to measure these parameters are CDC light traps and BG-Sentinel traps. Based on mosquito behavior, such as their attraction to pheromones or light, these traps use different attractants, such as light, CO₂, or human odor, to capture them. To estimate the population size and the ratio of sterile to fertile mosquitoes a common technique is to do Mark-release-recapture (MRR) studies. It consists in marking a subset of the released mosquitoes with a unique identifier and releasing them into the wild. By comparing the number of marked and unmarked mosquitoes captured in the traps, an estimate of the total population size and the ratio of sterile to fertile mosquitoes can be obtained. Some oviposition traps may be used to capture and count the number of eggs laid by female mosquitoes. To take into account the possible difficulty and cost of measuring all the variables (E, F, M and M_s) in the field, in the next sections (2.3.2 and 2.3.3), we propose feedback laws depending on less variables.

2.3.2 Feedback laws depending only on the total number of male mosquitoes

Some recent adult traps are able to count automatically the number of male mosquitoes that are captured and, even in a more classic setting, there exist traps that use synthetic versions of female insect pheromones to attract and capture male insects. This kind of traps placed at different locations in the field, allow us to determine $M + M_s$ of the target pest population. Our aim in this section is to build a feedback linearly depending on $M + M_s$. Consider the closed-loop system

$$\dot{z} = F(z, u(z)), \quad z = (E, M, F, M_s)^T \in \mathcal{D}', \tag{2.116}$$

where

$$u(z) = k(M + M_s), \quad (2.117)$$

$$F(z, u) = \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E \\ (1 - \nu) \nu_E E - \delta_M M \\ \nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \\ u - \delta_s M_s \end{pmatrix}, \quad (2.118)$$

and k is a fixed real number. Throughout all this section 2.3.2, we assume that (2.55) holds and that

$$k \in [0, \delta_s). \quad (2.119)$$

The offspring number related to this system is

$$\mathcal{R}_1(k) := \frac{(\delta_s - k) \beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\delta_s - (1 - \gamma_s) k)}. \quad (2.120)$$

Equilibria of the closed-loop system

Equilibria of the SIT model (2.116) are obtained by solving the system

$$\begin{cases} \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E = 0, \\ (1 - \nu) \nu_E E - \delta_M M = 0, \\ \nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F = 0, \\ kM - (\delta_s - k) M_s = 0. \end{cases} \quad (2.121)$$

We get either the extinction equilibrium $\mathbf{0}$, i.e.

$$E = 0, \quad M = 0, \quad F = 0, \quad M_s = 0 \quad (2.122)$$

or

$$\begin{aligned} E^* &= K \left(1 - \frac{1}{\mathcal{R}_1(k)}\right), \\ M^* &= \frac{(1 - \nu) \nu_E}{\delta_M} E^*, \\ F^* &= \frac{(\delta_s - k) \nu \nu_E}{\delta_F ((\delta_s - k) + \gamma_s k)} E^*, \\ M_s^* &= \frac{(1 - \nu) \nu_E k}{(\delta_s - k) \delta_M} E^*. \end{aligned} \quad (2.123)$$

Let us assume in the sequel that

$$\mathcal{R}_1(k) < 1. \quad (2.124)$$

Using (2.123) and (2.124), one gets $E^* < 0$ and therefore the equilibrium given by (2.123) is not relevant. In conclusion the closed-loop system (2.116) has one and only one equilibrium which is the extinction equilibrium $\mathbf{0}$. It is therefore tempting to raise the following conjecture (compare with Theorem 2.2.1).

Conjecture 2.3.1. *The extinction equilibrium $\mathbf{0}$ is globally asymptotically stable in $\mathcal{D}'$ for the closed-loop system (2.116).*

We have not been able to prove this conjecture. However,

1. In section 2.3.2, we give a positively invariant set for the closed-loop system (2.116) in which, as proved in section 2.3.2, $\mathbf{0}$ is globally asymptotically stable for (2.116);
2. In section 2.3.2, we provide numerical evidence for this conjecture.

Invariant set of the closed-loop system

From (2.55), (2.119), and (2.124), one gets

$$\frac{\beta_E \nu \nu_E - (\nu_E + \delta_E) \delta_F}{\beta_E \nu \nu_E - (1 - \gamma_s)(\nu_E + \delta_E) \delta_F} \delta_s < k < \delta_s. \quad (2.125)$$

Let us define, with $z = (E, M, F, M_s)^T$,

$$\mathcal{T}_1 := \{z \in \mathcal{D}' : \beta_E F(1 - \frac{E}{K}) \leq (\nu_E + \delta_E)E\}, \quad (2.126)$$

$$\mathcal{T}_3 := \{z \in \mathcal{D}' : (1 - \nu)\nu_E E \leq \delta_M M\}, \quad (2.127)$$

and, for $\kappa > 0$,

$$\mathcal{T}_2(\kappa) = \{z \in \mathcal{D}' : M \leq \kappa M_s\}. \quad (2.128)$$

One has the following theorem.

Theorem 2.3.3. *Assume that (2.125) holds and that*

$$\kappa \leq \frac{\gamma_s \delta_F (\nu_E + \delta_E)}{\beta_E \nu \nu_E - \delta_F (\nu_E + \delta_E)}, \quad (2.129)$$

$$\kappa \geq \frac{\delta_s - k}{k}. \quad (2.130)$$

Then $\mathcal{M}(\kappa) := \mathcal{T}_1 \cap \mathcal{T}_2(\kappa) \cap \mathcal{T}_3$ is a positively invariant set of the closed-loop system (2.116).

Remark 2.3.2. *Note that (2.125) implies that*

$$0 < \frac{\delta_s - k}{k} < \frac{\gamma_s \delta_F (\nu_E + \delta_E)}{\beta_E \nu \nu_E - \delta_F (\nu_E + \delta_E)}. \quad (2.131)$$

Hence there are $\kappa > 0$ such that both (2.129) and (2.130) hold.

Proof of Theorem 2.3.3. Let us first study the case where one starts with $E = F = 0$: we consider the Filippov solution(s) to the Cauchy problem

$$\dot{z} = F(z, u(z)), \quad E(0) = 0, \quad M(0) = M_0, \quad F(0) = 0, \quad M_s(0) = M_{s0}, \quad (2.132)$$

where $(M_0, M_{s0})^T \in [0, +\infty)^2$ is such that

$$M_0 \leq \kappa M_{s0}. \quad (2.133)$$

From (2.117), (2.118), and (2.132), one gets

$$E(t) = F(t) = 0, \quad \forall t \geq 0, \quad (2.134)$$

$$\dot{M} = -\delta_M M \text{ and } \dot{M}_s = kM - (\delta_s - k)M_s. \quad (2.135)$$

In particular, for every $t \geq 0$, $z(t) \in \mathcal{T}_1 \cap \mathcal{T}_3$. It remains to check that

$$z(t) \in \mathcal{T}_2(\kappa) \quad \forall t \geq 0. \quad (2.136)$$

From (2.135), one has

$$\frac{d}{dt} (M - \kappa M_s) = -(\delta_M + \kappa k)(M - \kappa M_s) - \kappa((1 + \kappa)k - \delta_s + \delta_M)M_s. \quad (2.137)$$

From (2.130) one has

$$(1 + \kappa)k - \delta_s + \delta_M \geq \delta_M. \quad (2.138)$$

Property (2.136) readily follows from (2.133), (2.137) and (2.138).

Let us now deal with the case where $E + F > 0$. Note that, for $z \in \mathcal{M}(\kappa)$, this implies that

$$E > 0 \text{ and } M > 0. \quad (2.139)$$

Until the end of the proof of Theorem 2.3.3 we assume that $z \in \mathcal{D}'$ and is such that (2.139) holds.

Let $h_1 : \mathcal{D}' \rightarrow \mathbb{R}$ be defined by

$$h_1(z) := \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E. \quad (2.140)$$

Its time derivative along the solution of the closed-loop system (2.116) is

$$\begin{aligned} \dot{h}_1(z) = & \beta_E \nu \nu_E E \frac{M}{M + \gamma_s M_s} (1 - \frac{E}{K}) \\ & - \delta_F \beta_E F(1 - \frac{E}{K}) - \frac{\beta_E^2 F^2}{K} (1 - \frac{E}{K}) \\ & + \frac{\beta_E (\nu_E + \delta_E) E F}{K} - (\nu_E + \delta_E) \beta_E F(1 - \frac{E}{K}) + (\nu_E + \delta_E)^2 E. \end{aligned} \quad (2.141)$$

For a set $\Sigma \subset \mathcal{D}'$, let us denote by $\partial\Sigma$ its boundary in $\mathcal{D}'$. On $\partial\mathcal{T}_1$,

$$\beta_E F(1 - E/K) = (\nu_E + \delta_E)E$$

. Hence

$$\dot{h}_1(z) = \beta_E \nu \nu_E E \frac{M}{M + \gamma_s M_s} (1 - \frac{E}{K}) - \delta_F (\nu_E + \delta_E) E \text{ if } z \in \partial\mathcal{T}_1. \quad (2.142)$$

In particular, using (2.129),

$$\dot{h}_1(z) \leq -\beta_E \nu \nu_E \frac{M}{M + \gamma_s M_s} \frac{E^2}{K} < 0 \text{ if } z \in \partial\mathcal{T}_1 \cap \mathcal{T}_2(\kappa). \quad (2.143)$$

Let us now turn to the behavior of the closed-loop system on the $\partial\mathcal{T}_2(\kappa)$. Let

$$h_2 : \mathcal{D}' \rightarrow \mathbb{R}$$

be defined by

$$h_2(z) := M - \kappa M_s. \quad (2.144)$$

Its time derivative along the solution of the closed-loop system (2.116) is

$$\dot{h}_2(z) = (1 - \nu) \nu_E E - \delta_M M - \kappa (kM - (\delta_s - k) M_s), \quad (2.145)$$

which leads to

$$\dot{h}_2(z) = (1 - \nu) \nu_E E - ((1 + \kappa)k - \delta_s + \delta_M) M \text{ if } z \in \partial\mathcal{T}_2(\kappa). \quad (2.146)$$

From (2.127), (2.138), and (2.146), one gets that

$$\dot{h}_2(z) \leq 0 \text{ if } z \in \mathcal{T}_3 \cap \partial\mathcal{T}_2(\kappa). \quad (2.147)$$

Finally, let us study the behavior of the closed-loop system on the $\partial\mathcal{T}_3$. Let

$$h_3 : \mathcal{D}' \rightarrow \mathbb{R}$$

be defined by

$$h_3(z) := (1 - \nu) \nu_E E - \delta_M M. \quad (2.148)$$

Its time derivative along the solution of the closed-loop system (2.116) is

$$\dot{h}_3(z) = \beta_E F \left(1 - \frac{E}{K} \right) - (\nu_E + \delta_E) E - \delta_M ((1 - \nu) \nu_E - \delta_M M), \quad (2.149)$$

which leads to

$$\dot{h}_3(z) = \beta_E F \left(1 - \frac{E}{K} \right) - (\nu_E + \delta_E) E \text{ if } z \in \partial\mathcal{T}_3. \quad (2.150)$$

In particular,

$$\dot{h}_3(z) \leq -\beta_E \frac{EF}{K} \leq 0 \text{ if } z \in \mathcal{T}_2(\kappa) \cap \partial\mathcal{T}_3. \quad (2.151)$$

This concludes the proof of Theorem 2.3.3. $\square$

Global asymptotic stability result

Let

$$\bar{\kappa} := \frac{\gamma_s \delta_F (\nu_E + \delta_E)}{\beta_E \nu \nu_E - \delta_F (\nu_E + \delta_E)}, \quad (2.152)$$

$$\mathcal{M} := \mathcal{M}(\bar{\kappa}). \quad (2.153)$$

Let us recall that, by (2.131), $\bar{\kappa}$, which clearly satisfies (2.129), satisfies also (2.130). In particular, by Theorem 2.3.3, $\mathcal{M}$ is positively invariant for the closed-loop system (2.116). The main result of this section is the following theorem.

Theorem 2.3.4. *Assume that (2.125) holds. Then $\mathbf{0}$ is globally asymptotically stable for the closed-loop system (2.116) in $\mathcal{M}$.*

Proof. The first step of the proof is the following lemma which shows that Theorem 2.3.4 holds with $\mathcal{M}$ replaced by $\mathcal{M}(\kappa)$ provided that (2.129) is a strict inequality and that (2.130) holds.

Lemma 2.3.1. *Let us assume that (2.130) holds and that*

$$\kappa < \frac{\gamma_s \delta_F (\nu_E + \delta_E)}{\beta_E \nu \nu_E - \delta_F (\nu_E + \delta_E)}. \quad (2.154)$$

Then $\mathbf{0}$ is globally asymptotically stable for system (2.116) in $\mathcal{M}(\kappa)$.

To prove this lemma we use a Lyapunov approach. Our Lyapunov function is $U : \mathcal{D}' \rightarrow \mathbb{R}_+$, $z \mapsto U(z)$,

$$U(z) = \delta_F E + \varepsilon M + \beta_E (1 + \varepsilon) F + \varepsilon^2 M_s, \quad (2.155)$$

where $\varepsilon \in (0, 1]$ is a constant which will be chosen later on. One has

$$U \text{ is of class } \mathcal{C}^1, \quad (2.156)$$

$$U(z) > U(\mathbf{0}) = 0, \quad \forall z \in \mathcal{D}' \setminus \{\mathbf{0}\}, \quad (2.157)$$

$$U(z) \rightarrow +\infty \text{ as } |z| \rightarrow +\infty \text{ with } z \in \mathcal{D}'. \quad (2.158)$$

Let us assume for the time being that

$$M + M_s \neq 0. \quad (2.159)$$

Then, the time derivative of U along the solution of the closed-loop system (2.116) is

$$\begin{aligned} \dot{U}(z) = & \delta_F \left(\beta_E F \left(1 - \frac{E}{K} \right) - (\nu_E + \delta_E) E \right) + \varepsilon ((1 - \nu) \nu_E E - \delta_M M) \\ & + \beta_E (1 + \varepsilon) \left(\nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \right) + \varepsilon^2 (kM - (\delta_s - k)M_s). \end{aligned} \quad (2.160)$$

In particular,

$$\begin{aligned} \dot{U}(z) \leq & -\varepsilon \delta_F \beta_E F \\ & - \left((\nu_E + \delta_E) - \varepsilon (1 - \nu) \nu_E - \beta_E (1 + \varepsilon) \nu \nu_E \frac{\kappa}{\kappa + \gamma_s} \right) E \\ & - \varepsilon (\delta_M - \varepsilon k) M - \varepsilon^2 (\delta_s - k) M_s \text{ if } z \in \mathcal{M}(\kappa). \end{aligned} \quad (2.161)$$

Let us now point out that (2.154) implies that

$$\beta_E \nu \nu_E \frac{\kappa}{\kappa + \gamma_s} < (\nu_E + \delta_E). \quad (2.162)$$

From (2.161) and (2.162) one gets that for $\varepsilon > 0$ small enough there exists $c(\varepsilon) > 0$ independent of $z \in \mathcal{M}(\kappa)$ such that

$$\dot{U}(z) \leq -c(\varepsilon) U(z) \text{ if } z \in \mathcal{M}(\kappa). \quad (2.163)$$

It remains to remove assumption (2.159). Let

$$t \mapsto z(t) = (E(t), M(t), F(t), M_s(t))^T$$

be a Filippov solution of the closed loop system for the initial condition $z(0) = (E_0, M_0, F_0, M_{s0})^T \in \mathcal{M}(\kappa)$. We observe that if $(E_0, F_0) = (0, 0)$, then $z(0) \in \mathcal{M}(\kappa)$ implies that $M_0 > 0$, from which one gets that $M(t) > 0$ for every $t \geq 0$. Hence (2.163) holds for every $t \geq 0$. While, if $(E_0, F_0) \neq (0, 0)$, then $M(t) > 0$ for every $t > 0$. In particular, one still has (2.161) and therefore (2.163) for every $t > 0$. Hence,

$$U(z(t)) \leq e^{-c(\varepsilon)t} U(z(0)), \quad \forall t \geq 0, \quad (2.164)$$

which, together with (2.157) and (2.158), concludes the proof of Lemma 2.3.1.

Let us now deduce from Lemma 2.3.1 that

$$\mathbf{0} \text{ is a global attractor for the closed-loop system (2.116) in } \mathcal{M}. \quad (2.165)$$

Let $z(t) = (E(t), M(t), F(t), M_s(t))^T$ be a Filippov solution of the closed-loop system (2.116) for the initial condition $z(0) = (E_0, M_0, F_0, M_{s0})^T \in \mathcal{M}(\kappa)$. If $(E_0, F_0) = (0, 0)$ then one has (2.134) and (2.135) which leads to $z(t) \rightarrow \mathbf{0}$ as $t \rightarrow +\infty$ (note that, by (2.131), $\delta_s - k > 0$). Let $h_2 : \mathcal{D}' \rightarrow \mathbb{R}$ be defined by

$$\bar{h}_2(z) := M - \bar{\kappa} M_s. \quad (2.166)$$

Note that, if for some $t_0 \geq 0$, $\bar{h}_2(z(t_0)) < 0$, then there exists $\kappa > 0$ satisfying (2.130) and (2.154) such that $z(t_0) \in \mathcal{M}(\kappa)$. By Lemma 2.3.1 one then has $z(t) \rightarrow \mathbf{0}$ as $t \rightarrow +\infty$. If there is no such t_0 , then

$$\bar{h}_2(z(t)) = 0 \text{ for every } t \geq 0. \quad (2.167)$$

From (2.146) with $\kappa = \bar{\kappa}$, (2.152), (2.166), and (2.167), one gets that

$$h_3(z(t)) = 0 \text{ for every } t \geq 0, \quad (2.168)$$

which together with (2.151) implies that

$$E(t)F(t) = 0 \text{ for every } t \geq 0. \quad (2.169)$$

Since $z(t) \in \mathcal{T}_1$, (2.126) and (2.169) imply that

$$F(t) = 0 \text{ for every } t \geq 0. \quad (2.170)$$

Then, if for some $t_0 \geq 0$, $E(t_0) = 0$, one has $(E(t_0), F(t_0)) = (0, 0)$, which, as already pointed out above, implies that $z(t) \rightarrow \mathbf{0}$ as $t \rightarrow +\infty$. It remains to handle the case where

$$E(t) > 0 \text{ for every } t \geq 0. \quad (2.171)$$

In particular, since $z(t) \in \mathcal{T}_3$, one has, using (2.127),

$$M(t) > 0 \text{ for every } t \geq 0. \quad (2.172)$$

Then, differentiating (2.170) with respect to time and using (2.116) and (2.118), one gets

$$E(t) = 0 \text{ for every } t \geq 0, \quad (2.173)$$

which leads to a contradiction with (2.171). This concludes the proof of (2.165).

In order to conclude the proof of Theorem 2.3.4 it just remains to check that

$$\mathbf{0} \text{ is stable for the closed-loop system (2.116) in } \mathcal{M}. \quad (2.174)$$

For that, let $\bar{U} : \mathcal{D}' \rightarrow \mathbb{R}_+$, $z \mapsto \bar{U}(z)$, be defined by

$$\bar{U}(z) = \delta_F E + \beta_E F, \quad (2.175)$$

which corresponds to the definition of U given in (2.155) with $\varepsilon = 0$. Let

$$z(t) = (E(t), M(t), F(t), M_s(t))^T$$

be a Filippov solution of the closed loop system for the initial condition $z(0) = (E_0, M_0, F_0, M_{s0})^T \in \mathcal{M}$. As above, we may restrict our attention to the case where

$$E(t) > 0 \text{ for every } t > 0. \quad (2.176)$$

Let us recall that since $z(t) \in \mathcal{M} \subset \mathcal{T}_3$, (2.127), and (2.176) imply that

$$M(t) > 0 \text{ for every } t \geq 0. \quad (2.177)$$

Then, $\bar{U}(z(t))$ can be differentiated with respect to time and one has, by (2.161) with $\varepsilon = 0$ and $\kappa = \bar{\kappa}$, and (2.152),

$$\dot{\bar{U}}(z(t)) \leq 0, \quad (2.178)$$

which shows that

$$E(t) + F(t) \leq \frac{\max\{\delta_E, \delta_F\}}{\min\{\delta_E, \delta_F\}} (E(0) + F(0)), \text{ for every } t \geq 0. \quad (2.179)$$

It remains to estimate $M(t)$ and $M_s(t)$. Using $z(t) \in \mathcal{T}_2(\bar{\kappa})$ and (2.128), one already has

$$M(t) \leq \bar{\kappa} M_s(t) \text{ for every } t \geq 0. \quad (2.180)$$

Using (2.116), (2.117), (2.118), (2.131), (2.152), and (2.180), one has

$$\dot{M}_s(t) \leq (k\bar{\kappa} + k - \delta_s) M_s(t) \leq 0 \text{ for every } t \geq 0. \quad (2.181)$$

In particular, using also (2.180),

$$M_s(t) \leq M_s(0) \text{ and } M(t) \leq \bar{\kappa} M_s(0) \text{ for every } t \geq 0. \quad (2.182)$$

This concludes the proof of (2.174) and, therefore, of Theorem 2.3.4. $\square$

Numerical simulations

In this section, we will show numerical simulations of the dynamics when we apply feedback (2.117). We fix $z_0 = (21910, 5587, 13419, 0) \notin \mathcal{M}$. We now compute condition (2.125) according to the parameter set in the table 2.1. This gives $0.11843 < k < 0.12$. We take $k = 0.119$. The following figures show the evolution of the states when condition (2.125) holds.

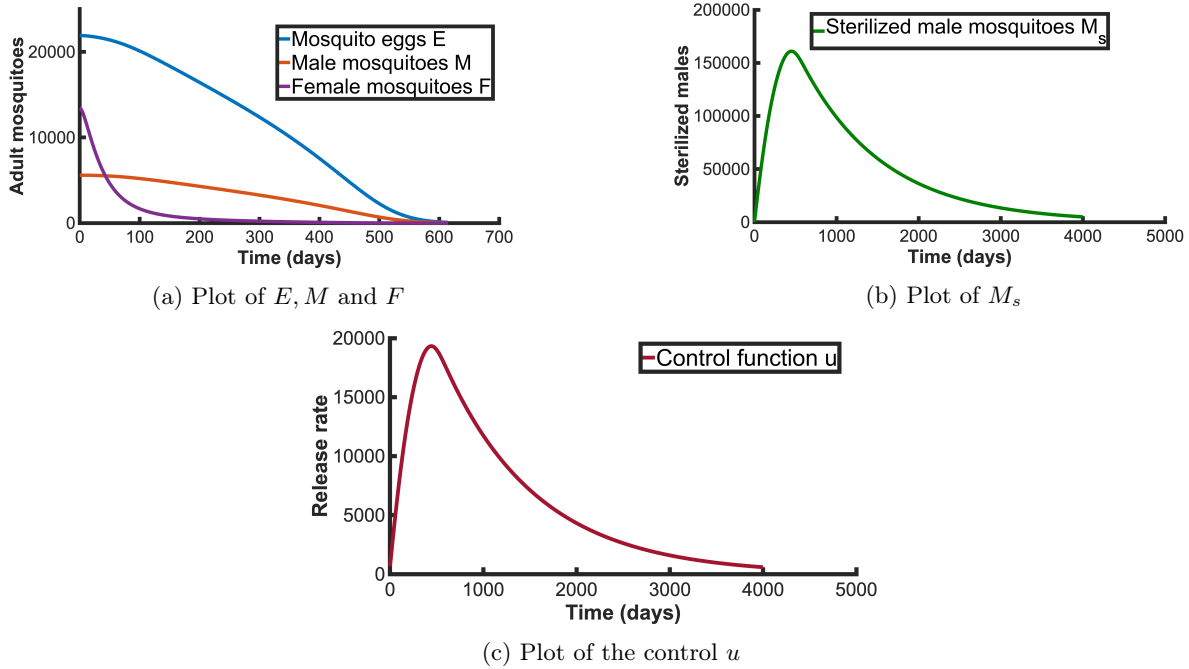


Figure 2.3.3. (a): Plot of E, M and F for system (2.43)-(2.46) when applying feedback (2.117). with the initial condition $z_0 \notin \mathcal{M}$ and final time $T = 800$. (b): Plot of M_s for final time $T = 4000$ when we apply the backstepping feedback (2.117). (c): Plot of the feedback control function (2.117).

Remark 2.3.3. We observe that the convergence time of the states E, M and F is longer than when we applied the backstepping feedback control (2.94). In this case, with $t_f = 700$ days,

$$\int_0^{t_f} u(t) dt \approx 17,91 \text{ millions.} \quad (2.183)$$

We take several initial conditions randomly and plot the resulting dynamics in figure 2.3.4,

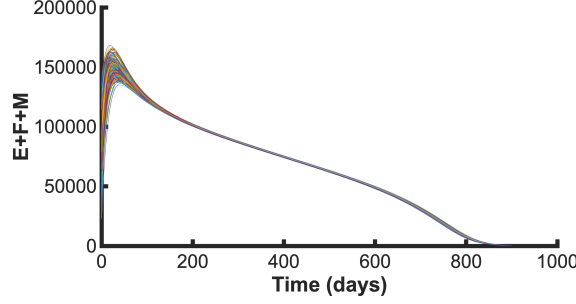


Figure 2.3.4. Plot of $\|x(x_0, t)\|_1$ when applying the feedback (2.117) with several randomly chosen initial conditions x_0 .

Robustness test

To analyze the robustness of our feedback against variations of the parameters, we carry out some variation of the parameters (new values) in table 2.2. The results are summarized in table 2.2. We observe that very small

Old parameters	New Parameters	Simulation
• $\nu_E = 0.05$ • $\delta_E = 0.03$ • $\delta_F = 0.04$ • $\delta_M = 0.1$ • $\delta_s = 0.12$ • $\beta_E = 8$	• $\nu_E = 0.08$ • $\delta_E = 0.046$ • $\delta_F = 0.033$ • $\delta_M = 0.12$ • $\delta_s = 0.139$ • $\beta_E = 11$	• Plot of E, M and F

Table 2.2: Robustness test

perturbations of the parameters destabilize the origin.

2.3.3 Feedback laws depending only on wild male mosquitoes

In the application of the technique it might also be possible to estimate only fertile males. For instance, in MRR experiments, sterile mosquitoes are identified by the presence of a marker, such as a dye or a fluorescent protein, which has been applied before their release (although, at present, it is not always easy to do this for all the mosquitoes released in field interventions). Nevertheless, since the technology is evolving very fast, it is possible that it can become standard practice in the near future (for instance, we recall that PCR analysis of the captured mosquitoes is already currently used thanks to genetic bar-coding). Thus, it is interesting to set up the mathematical techniques to deal with this situation. Therefore, we consider in this section the case where the feedback depends only on the state M . Consider the closed-loop system

$$\dot{z} = F(z, u(z)), \quad z = (E, M, F, M_s)^T \in \mathcal{D}', \quad (2.184)$$

where

$$u(z) = \lambda M \quad (2.185)$$

and

$$F(z, u(z)) = \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E \\ (1 - \nu) \nu_E E - \delta_M M \\ \nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \\ \lambda M - \delta_s M_s \end{pmatrix}, \quad (2.186)$$

The offspring number related to this system is

$$\mathcal{R}_2(\lambda) := \frac{\delta_s \beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\delta_s + \gamma_s \lambda)}. \quad (2.187)$$

We assume that

$$\mathcal{R}_2(\lambda) < 1. \quad (2.188)$$

Note that this inequality is equivalent to

$$\lambda > \frac{(\beta_E \nu \nu_E - (\nu_E + \delta_E) \delta_F) \delta_s}{\gamma_s (\nu_E + \delta_E) \delta_F}. \quad (2.189)$$

Let us point out that the closed-loop system (2.184) is exactly the closed-loop system (2.116) if one performs the following change of variables (with natural notations):

$$k^{(2.116)} = \lambda^{(2.184)} \text{ and } \delta_s^{(2.116)} = \delta_s^{(2.184)} + \lambda^{(2.184)}. \quad (2.190)$$

Hence Theorem 2.3.3 and Theorem 2.3.4 lead to the following theorem.

Theorem 2.3.5. *Assume that (2.55) and (2.189) hold. Then $\mathcal{M}$ is positively invariant for the closed-loop system (2.184) and $\mathbf{0}$ is globally asymptotically stable for the closed-loop system (2.184) in $\mathcal{M}$.*

Numerical simulations

In this section, we present the numerical evolution of the states when we apply feedback (2.117). We fix as initial condition $z_0 = (21910, 5587, 13419, 0) \notin \mathcal{M}$ and $K = 22200 \text{ ha}^{-1}$. We now compute condition (2.125) according to the parameters set in table 2.1. This gives $\lambda > 9.06$. We take for the simulation $\lambda = 22$.

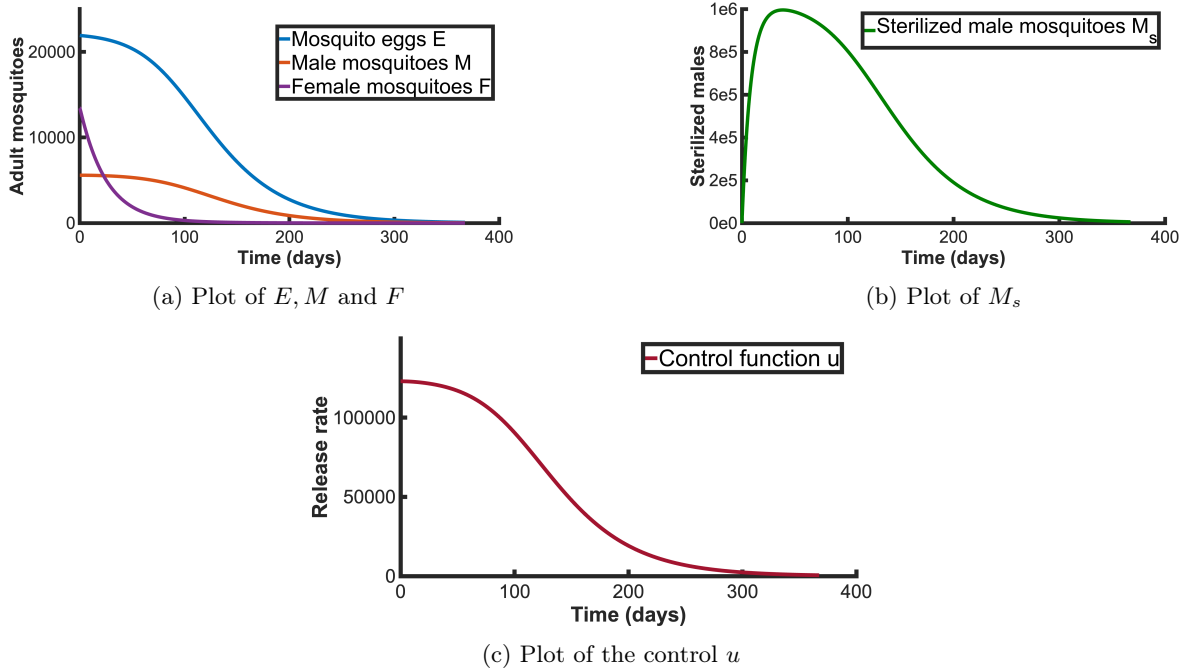


Figure 2.3.5. (a): The results of the simulation E , M and F for system (2.43)-(2.46) when applying the feedback (2.185) with the initial condition $z_0 \notin \mathcal{M}$ for final time $T = 400$ and $\lambda = 22$. (b): Plot of M_s for final time $T = 400$. (c): Plot of the control function (2.185).

Remark 2.3.4. Notice that with $t_f = 400$ days,

$$\int_0^{t_f} u(t) dt \approx 17.28 \text{ millions.} \quad (2.191)$$

In figure 2.3.6 we take several initial conditions randomly for $\lambda = 22$.

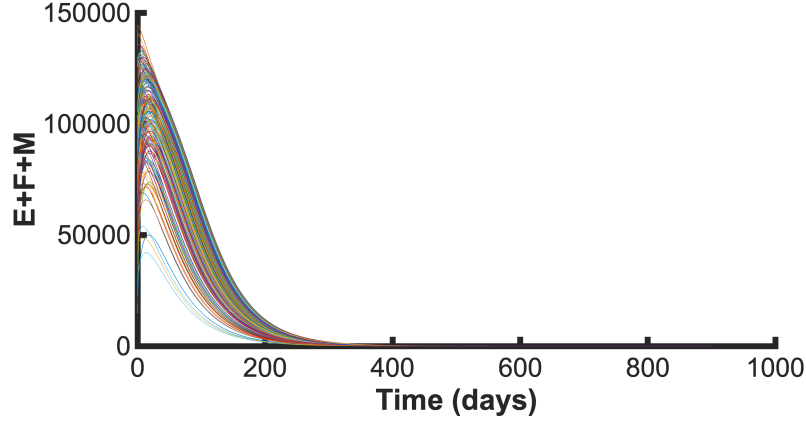


Figure 2.3.6. Plot of $\|x(x_0, t)\|_1$ when applying the feedback (2.185) with several randomly chosen initial conditions x_0 .

Robustness test

We test the robustness using the same protocol as in section 2.3.1. Figure 2.3.7 shows the results for 200 randomly chosen initial conditions in $[0, 10K]^4$.

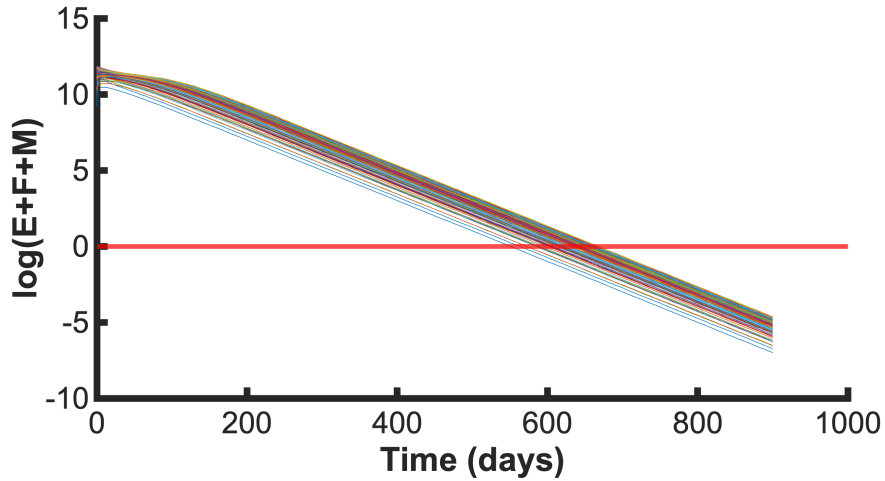


Figure 2.3.7. Robustness test when applying the feedback law (2.185) with $\lambda = 22$.

We observe that feedback (2.185) is robust with respect to changes of parameters: for rather large perturbations on the parameters it stills globally stabilizes the dynamics at the extinction equilibrium.

2.4 Comparison of the feedback laws

In this section, we use numerical simulations to carry out a comparative study of the feedback control (2.94) and (2.185). We consider that the environmental capacity $K = 22200 \text{ ha}^{-1}$ and that the initial condition is the persistence equilibrium. Our comparison criteria are the intervention time and the control cost obtained when

applying the different feedback laws. The results are presented in Tables 2.3 and 2.4 where the intervention is presented until $E = \frac{K}{100}$. λ is used to regulate the control feedback law (2.185) while the regulation parameters for the backstepping control are α , θ and $\beta_s = 1 \text{ Day}^{-1}$.

Table 2.3 shows the intervention time and control cost for different values of λ . In table 2.4, we fix $\alpha = 80$ and present the results obtained for different values of θ . Note that since α and θ are regulatory values for control (2.94), a study can be carried out to find their optimal values in order to have a better value of the control (2.94) presented here.

u_λ Intervention														
λ	9.06	10	11	12	13	14	15	16	17	18	19	20	21	22
$T_\lambda \text{ (Day}^{-1}\text{)}$	667	477	424	390	367	350	336	326	318	311	305	300	295	291
$(\int_0^T u_\lambda)$	8.24e6	8.61e6	9.14e6	9.72e6	1.03e7	1.09e7	1.16e7	1.22e7	1.29e7	1.35e7	1.42e7	1.48e7	1.55e7	1.61e7

Table 2.3: Intervention time and control cost for different values of λ

u_θ Intervention													
θ	100	110	120	130	140	150	160	170	180	190	200	210	220
$T_\theta \text{ (Day}^{-1}\text{)}$	484	445	417	396	379	366	355	345	338	331	325	319	315
$(\int_0^T u_\theta)$	6.49e6	6.74e6	7.02e6	7.33e6	7.65e6	7.98e6	8.32e6	8.67e6	9.02e6	9.38e6	9.74e6	1.01e7	1.04e7

Table 2.4: Intervention time and control cost for different values of θ

Remark 2.4.1. For $\lambda = 10$ and for $\theta = 170$ we obtain nearly the same control cost for the two different interventions, but the convergence time for the u_θ intervention is smaller. This means that for the same control cost, the u_θ intervention saves time.

For $(\lambda, \theta) = (13, 150)$ and $(17, 210)$, the two interventions give approximately the same convergence time, but the cost is least for the u_θ intervention. We conclude that for the same convergence times, the u_θ intervention offers a better cost.

In conclusion, we note that thanks to the α and θ control parameters, despite the non-linearity of the backstepping control, it offers a better result in terms of both convergence time and control cost.

2.5 Conclusion

We have built feedback laws that stabilize the SIT dynamical model and have studied their robustness with respect to changes of parameters. We study three types of feedback laws:

- 1) a backstepping one in section 2.3.1.
- 2) one depending linearly on the total number of male mosquitoes, $M + M_s$ in section 2.3.2.
- 3) one depending linearly on the number of wild male mosquitoes M in section 2.3.3.

For the first one we were able to prove the global asymptotic stability. Based on the analysis done in section 2.4 we see that this feedback law gives a better result in terms of both convergence time and control cost. However, it depends on three variables (E , M and M_s) which may be difficult to measure in the field.

For the second one, we proved the global asymptotic stability only in a certain invariant set $\mathcal{M}$. We conjecture that this feedback gives global stability and we show numerical evidence for this conjecture (see figure 2.3.4). The advantage of this feedback law is that it depends only on the total number of male mosquitoes $M + M_s$ which is a natural quantity to measure in the field.

However, this feedback law has an important drawback due to the narrow interval allowed for the gain α of the feedback in (2.125). This might pose a problem for the robustness of this method relative to the variations of the biological parameters.

For the third one, we proved the global asymptotic stability only in a certain invariant set $\mathcal{M}$. We also conjecture that this feedback gives global stability and we show numerical evidence for this conjecture (see figure 2.3.6). The main difference w.r.t. the previous feedback law is that now the method is robust w.r.t. variations of the biological parameters. However, the drawback in this case is that M should be harder to measure in the field.

Changes of the environment in time, and in particular seasons (both in tropical and in temperate climates), are known to have a big impact on the mosquito populations and it will thus be important to take them into account in our future work.

Also in our work, we did not consider the pest population's spatial distribution. This has again an impact in practical terms and has been considered in several mathematical works and, in particular, those concerning invasion wave blocking [13], the rolling carpet strategy [14] or a space dependent mosquito carrying capacity [9].

In our future works, we will construct observers that can estimate the state from easily measurable variables (after this paper was submitted a first observer construction was done in [1]) and we will also integrate the spatial aspect in this dynamical model. After the first version of this paper, other output feedback results using reinforcement learning were obtained in [4, 3].

As stated in the introduction, although the paper is mostly written for the specific case of mosquitoes, our results can be extended to the case of other pests for which the Sterile Insect Technique is pertinent.

Acknowledgements

The authors would like to thank Hervé Bossin and René Gato for the very interesting discussions that helped them identify feedback laws that can be useful for field applications and to be aware of their limitations. We hope that our future collaborations will allow us to develop and apply the ideas put forward in this work in field interventions and learn from the results to be able to improve our strategies.

Chapter 3

Feedback stabilization and observer design for sterile insect technique model

This chapter was published as an article in Mathematical Biosciences and Engineering [1].

Abstract. This paper focuses on the feedback global stabilization and observer construction for a sterile insect technique model. The Sterile Insect Technique (SIT) is one of the most ecological methods for controlling insect pests responsible for worldwide crop destruction and disease transmission.

In this work, we construct a feedback law that globally asymptotically stabilizes a SIT model at extinction equilibrium. Since the application of this type of control requires the measurement of different states of the target insect population, and, in practice, some states are more difficult or more expensive to measure than others, it is important to know how to construct a state estimator which from a few well-chosen measured states, estimates the other ones, as the one we build in the second part of our work. In the last part of our work, we show that we can apply the feedback control with estimated states to stabilize the full system.

3.1 Introduction

The Sterile Insect Technique, or SIT, is presently one of the most ecological methods for controlling insect pests responsible for disease transmission or crop destruction worldwide. This technique consists in releasing sterile males into the insect pest population [23, 54, 110]. This approach aims at reducing fertility and, consequently, reducing the target insect population after a few generations. Classical SIT has been modeled and studied theoretically in a large number of papers to derive results to study the success of these strategies using discrete, continuous, or hybrid modeling approaches (for instance, the recent papers [102, 12, 13, 8, 74, 33, 32]).

Despite this extensive research, little has been done concerning the stabilization of the target population near extinction after the decay caused by the massive initial SIT intervention and there are still major difficulties due to the complexity of the dependency on climate, landscape and many other parameters which would be difficult to be integrated into the mathematical models studied. Not being able to consider all these parameters in our mathematical models and knowing that these external factors strongly impact the evolution of the density of the target population, we focus our studies on releases that now depend on the target population density measurements since, as we will see below, this makes our control more robust. Indeed, several monitoring tools can provide information on the size of the wild population throughout the year. So, a control that considers this information to adapt the size of the releases is possible and useful. This was already the case of [2, 33] in which a state feedback control law gives significant robustness qualities to the mathematical model of SIT. Although this approach provides evidence in terms of robustness because the control is directly adjusted according to the density of the population, its application requires to continuously measure the different states of the model. In practice, traps allow data to be collected to analyze the control's impact and technology is being developed that may allow us to obtain continuous data in the near future.

However, specific categories of data are still problematic or very expensive to obtain. For example, during a SIT intervention, it is difficult to measure the density of young females that have not yet been fecundated or of females that were fecundated by wild males. In this work we use another control theory tool, which consists of constructing a state estimator for a dynamical system and using this estimator to apply feedback control. A state observer or state estimator is a system that provides an estimate of the natural state using some partial

measurements of the real system. In our case, using traps, wild males as well as sterile males, can be measured. Using the observer system technique, we have built a system that allows us to estimate all other states.

The problem of observer design for linear systems was established and solved by [62] and [79]. While Kalman's Observer [62] was highly successful for linear systems, extending it to nonlinear systems took a lot of work. In several cases, the observer can be obtained from the extended Kalman filter by a particular choice of the matrix gain using Linear Matrix Inequalities (LMIs). The development of the observer in this paper was motivated by its application to the SIT model. A model of this process can be written as

$$\dot{x} = Ax + B(y)x + Du, \quad (3.1)$$

$$y = Cx, \quad (3.2)$$

where $y \in \mathbb{R}^m$ is the output, $x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}^p$ is the input. The output matrix $B(y)$ is such that the coefficients $b(y)_{ij}$ are bounded for all i, j .

Our paper has three parts. In the first part, thanks to the backstepping approach, we build a feedback control law that stabilizes the zero population state for the SIT model for the mosquito population, which considers only the compartments of young females and fertilized females presented in [18]. In the second part we construct a state estimator for the SIT model. Finally, in the third part we show that the application of this feedback, depending on the measured states and the ones estimated thanks to the state estimator, globally stabilizes the system.

3.2 Mosquito Population dynamics

The mosquito life cycle has several phases. The aquatic stage comprises eggs, larvae, and pupa, followed by the adult stage, where we consider both wild males and females. After emergence from the pupa, a female mosquito needs to mate and then to get a blood meal before it can start laying eggs. Then every 4 – 5 days, it will take a blood meal and lay 100 – 150 eggs at different places (10 – 15 per place). For the mathematical description, we will consider the following compartments [18].

- E the density of population in aquatic stage,
- Y the density of young females, not yet laying eggs,
- F the density of fertilized and egg-laying females,
- M the density of males,
- M_s the density of sterile males,
- U the density of females that mate with sterile males.

The Y compartment represents the stage of the young females before the start of their gonotrophic cycle, i.e., before they mate and take their first blood meal. It generally lasts for 3 to 4 days. The sterile insect technique introduces male mosquitoes to compete with wild males. We denote by M_s the density of sterile mosquitoes and by U the density of females that have mated with them. We assume that a female mating mosquito has probability $\frac{M}{M+M_s}$ to mate with a wild male and probability $\frac{M_s}{M+M_s}$ to mate with a sterile one. Hence, the transfer rate η from the compartment Y splits into transfer rate of $\frac{\eta_1 M}{M+M_s}$ to compartment F and a transfer rate of $\frac{\eta_2 M_s}{M+M_s}$ to compartment U of females that will be laying sterile (nonhatching) eggs. The mathematical model is the system of ordinary differential equations presented in [50]

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\delta_E + \nu_E)E, \quad (3.3)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (3.4)$$

$$\dot{Y} = \nu\nu_E E - \frac{\eta_1 M}{M+M_s} Y - \frac{\eta_2 M_s}{M+M_s} Y - \delta_Y Y, \quad (3.5)$$

$$\dot{F} = \frac{\eta_1 M}{M+M_s} Y - \delta_F F, \quad (3.6)$$

$$\dot{U} = \frac{\eta_2 M_s}{M+M_s} Y - \delta_U U, \quad (3.7)$$

$$\dot{M}_s = u - \delta_s M_s. \quad (3.8)$$

The parameter δ_Y is the mortality rate, for young females (they can die without mating for diverse reason like predators or other hostile environmental conditions). Male mosquitoes can mate for most of their lives. A female mosquito needs a successful mating to be able to reproduce for the rest of her life, $\beta_E > 0$ is the oviposition rate; $\delta_E, \delta_M, \delta_F, \delta_Y, \delta_s > 0$ are the death rates respectively for eggs, wild adult males, fertilized females, young females and sterile males; $\nu_E > 0$ is the hatching rate for eggs; $\nu \in (0, 1)$, the probability that a pupa gives rise to a

female, and $(1 - \nu)$ is, therefore, the probability of giving rise to a male. $K > 0$ is the environmental capacity for eggs. It can be interpreted as the maximum density of eggs that females can lay in breeding sites. Since here the larval and pupal compartments are not present, we consider that E represents all the aquatic compartments, in which case, this term K represents a logistic law's carrying capacity for the aquatic phase, which also includes the effects of competition between larvae. The control function u represents the number of mosquitoes released during the SIT intervention. It is interesting to follow the evolution of the state U because female mosquitoes, once fertilized by sterile males, will continue their gonotrophic cycle normally and, therefore, can still transmit disease. We will assume in this work that

$$\delta_s \geq \delta_M. \quad (3.9)$$

In [18, 50], equilibria and their stability property were studied for the system without control.

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\delta_E + \nu_E)E, \quad (3.10)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (3.11)$$

$$\dot{Y} = \nu\nu_E E - (\eta_1 + \delta_Y)Y, \quad (3.12)$$

$$\dot{F} = \eta_1 Y - \delta_F F. \quad (3.13)$$

$$(3.14)$$

Its basic offspring number is $\mathcal{R}_0 = \frac{\eta_1 \beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\eta_1 + \delta_Y)}$. For the rest of our work, we assume that

$$\mathcal{R}_0 > 1. \quad (3.15)$$

3.3 Global stabilization by a feedback law

We assume that wild males are more likely to fertilize young females because they are born in the same egg-laying site. We define

$$\Delta\eta = \eta_1 - \eta_2 \geq 0. \quad (3.16)$$

Other authors, such as in [18], have already studied the stability of this type of model. The difference in our approach lies in the kind of control used initially for global stabilization. Indeed, in most of the prior studies the controls u studied were independent of system states. Some previous works have considered certain simple applications of feedback control to SIT (see, for instance, [33, 30, 46]). In a previous paper, [2], we used the backstepping method to build a feedback control system that simplifies the SIT model, which is presented in [102], assuming that all females are immediately fertilized. Here we consider the system

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\delta_E + \nu_E)E, \quad (3.17)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (3.18)$$

$$\dot{Y} = \nu\nu_E E - \frac{\Delta\eta M}{M + M_s} Y - (\eta_2 + \delta_Y)Y, \quad (3.19)$$

$$\dot{F} = \frac{\eta_1 M}{M + M_s} Y - \delta_F F, \quad (3.20)$$

$$\dot{U} = \frac{\eta_2 M_s}{M + M_s} Y - \delta_U U, \quad (3.21)$$

$$\dot{M}_s = u - \delta_s M_s. \quad (3.22)$$

Let $\mathcal{N} := [0, +\infty)^6$ and $\mathcal{X} := (E, M, Y, F, U, M_s)^T$. When applying a feedback law $u : \mathcal{N} \rightarrow [0, +\infty)$, the closed-loop system is the system

$$\dot{\mathcal{X}} = H(\mathcal{X}, u(\mathcal{X})), \quad (3.23)$$

where H is the right-hand side of equations (3.17)-(3.22). The construction method remains the same as in our previous paper [2]. In this work, we also consider solutions in the Filippov sense of our discontinuous closed-loop system (see, for instance [51, 60, 52, 45, 41, 22]). Let us define $x := (E, M, Y, F, U)^T$. We must rewrite the target system (3.17)-(3.21) in the following form to apply the backstepping method (see, for instance [44, Theorem 12.24, page 334]):

$$\begin{cases} \dot{x} = f(x, M_s), \\ \dot{M}_s = u - \delta_s M_s, \end{cases} \quad (3.24)$$

where $f : \mathbb{R}^6 \rightarrow \mathbb{R}^5$ represents the right hand side of (3.17)-(3.21). We then consider the control system $\dot{x} = f(x, M_s)$ with the state being $x \in \mathcal{D} := [0, +\infty)^5$ and the control being $M_s \in [0, +\infty)$. We assume that M_s is of the form $M_s = \theta M$ for a constant $\theta > 0$. Then, we define and study the closed-loop system

$$\dot{x} = f(x, \theta M). \quad (3.25)$$

Its offspring number is

$$\mathcal{R}(\theta) := \frac{\beta_E \eta_1 \nu \nu_E}{\delta_F(\nu_E + \delta_E)(\Delta\eta + (1 + \theta)(\eta_2 + \delta_Y))}. \quad (3.26)$$

Note that if $\mathcal{R}(\theta) \leq 1$, $\mathbf{0} \in \mathbb{R}^5$ is the only equilibrium point of the system in $\mathcal{D}$. Our next proposition shows that the feedback law $M_s = \theta M$ stabilizes our control system $\dot{x} = f(x, M_s)$ if $\mathcal{R}(\theta) < 1$.

Proposition 3.3.1. *Assume that*

$$\mathcal{R}(\theta) < 1. \quad (3.27)$$

Then $\mathbf{0}$ is globally exponentially stable in $\mathcal{D}$ for system (3.25). The exponential convergence rate is bounded from below by the positive constant c defined by relation (3.31).

Proof. We apply Lyapunov's second theorem. To do so, we define $V : [0, +\infty)^5 \rightarrow \mathbb{R}_+$, $x \mapsto V(x)$,

$$V(x) := \frac{(1 + 2\mathcal{R}(\theta))\nu\nu_E}{(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))}E + \nu M + \frac{3\mathcal{R}(\theta)}{(1 - \mathcal{R}(\theta))}Y + \frac{(2 + \mathcal{R}(\theta))\beta_E\nu\nu_E}{\delta_F(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))}F + \sigma U, \quad (3.28)$$

where $\sigma > 0$ is a constant, that we will choose later.

As (3.27) holds, V is of class $\mathcal{C}^1$, $V(x) > V((0, 0, 0, 0, 0)^T) = 0$, $\forall x \in [0, +\infty)^5 \setminus \{(0, 0, 0, 0, 0)^T\}$, $V(x) \rightarrow +\infty$ when $\|x\| \rightarrow +\infty$ with $x \in \mathcal{D}$ and

$$\begin{aligned} \dot{V}(x) = & -\frac{\beta_E\nu\nu_E}{(\nu_E + \delta_E)}F - \nu\delta_M M - \frac{(1 + 2\mathcal{R}(\theta))\nu\nu_E}{(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))} \frac{\beta_E}{K} F E \\ & - \nu^2\nu_E E - \frac{\eta_1\beta_E\nu\nu_E}{\delta_F(\nu_E + \delta_E)(1 + \theta)}Y - \frac{\sigma\eta_2}{1 + \theta}Y + \sigma\eta_2 Y - \sigma\delta_U U. \end{aligned}$$

By choosing

$$\sigma := \frac{\eta_1\beta_E\nu\nu_E\mathcal{R}(\theta)}{(1 + \theta)\eta_2(\nu_E + \delta_E)\delta_F} \quad (3.29)$$

we get

$$\begin{aligned} \dot{V}(x) = & -\frac{\beta_E\nu\nu_E}{(\nu_E + \delta_E)}F - \nu\delta_M M - \frac{(1 + 2\mathcal{R}(\theta))\nu\nu_E}{(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))} \frac{\beta_E}{K} F E \\ & - \nu^2\nu_E E - \frac{\eta_1\beta_E\nu\nu_E(1 + \theta(1 - \mathcal{R}(\theta)))}{\delta_F(\nu_E + \delta_E)(1 + \theta)^2}Y - \sigma\delta_U U. \end{aligned}$$

and using once more (3.27), we get

$$\dot{V}(x) \leq -cV(x), \quad \forall x \in [0, +\infty)^5, \quad (3.30)$$

with

$$c := \min \left\{ \frac{\nu(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))}{(1 + 2\mathcal{R}(\theta))}, \delta_M, \frac{\delta_F(1 - \mathcal{R}(\theta))}{2 + \mathcal{R}(\theta)}, \frac{\eta_1\beta_E\nu\nu_E(1 + \theta(1 - \mathcal{R}(\theta)))}{\delta_F(\nu_E + \delta_E)(1 + \theta)^2} \frac{(1 - \mathcal{R}(\theta))}{3\mathcal{R}(\theta)}, \delta_U \right\} > 0. \quad (3.31)$$

This concludes the proof of Proposition 3.3.1. $\square$

Remark 3.3.1. *When the Allee effect is included in the model (for instance [102, eq:2.5, Page 25]), the control $M_s = \theta M$ can still be used, and the proof of the stability result can still be done using the same Lyapunov function (3.28).*

We define

$$\phi := \frac{(2 + \mathcal{R}(\theta))\eta_1\beta_E\nu\nu_E - 3\mathcal{R}(\theta)\Delta\eta\delta_F(\nu_E + \delta_E)}{\delta_F(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))(1 + \theta)} - \frac{\eta_1\beta_E\nu\nu_E\mathcal{R}(\theta)}{(1 + \theta)^2(\delta_E + \nu_E)\delta_F}, \quad (3.32)$$

$$Q := 3(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E)\delta_F - (1 - \mathcal{R}(\theta))\eta_1\beta_E\nu\nu_E, \quad (3.33)$$

and for $\alpha > 0$, the map $G : \mathcal{N} := [0, +\infty)^6 \rightarrow \mathbb{R}$, $(x^T, M_s)^T \mapsto G((x^T, M_s)^T)$ by

$$G((x^T, M_s)^T) := \frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)(3\theta M + M_s)} + \frac{((1 - \nu)\nu_E\theta E - \theta\delta_M M)(\theta M + 3M_s)}{3\theta M + M_s} + \delta_s M_s + \frac{1}{\alpha}(\theta M - M_s) \text{ if } M + M_s \neq 0, \quad (3.34)$$

$$G((x^T, M_s)^T) := 0 \text{ if } M + M_s = 0. \quad (3.35)$$

Finally, let us define the feedback law $u : \mathcal{N} \rightarrow [0, +\infty)$, $(x^T, M_s)^T \mapsto u((x^T, M_s)^T)$, by

$$u((x^T, M_s)^T) := \max\left(0, G((x^T, M_s)^T)\right). \quad (3.36)$$

The global stability result is the following.

Theorem 3.3.1. *Assume that (3.27) holds. Then $\mathbf{0} \in \mathcal{N}$ is globally exponentially stable in $\mathcal{N}$ for system (3.17)-(3.21) with the feedback law (3.36). The exponential convergence rate is bounded by the positive constant c_p defined by*

$$c_p := \min\left\{c, \frac{1}{\alpha}, \delta_M, \frac{Q}{3(1 + \theta)\delta_F(\nu_E + \delta_E)}, \delta_U\right\}. \quad (3.37)$$

Lemma 3.3.1. *Assume that (3.15) and (3.27) hold, then $\phi > 0$.*

Proof.

Let us define $\phi_1 := \frac{(2 + \mathcal{R}(\theta))\eta_1\beta_E\nu\nu_E - 3\mathcal{R}(\theta)\Delta\eta\delta_F(\nu_E + \delta_E)}{\delta_F(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))(1 + \theta)}$. We get from the relation (3.15) that $\eta_1\beta_E\nu\nu_E > \delta_F(\nu_E + \delta_E)(\eta_1 + \delta_Y)$. So

$$\phi_1 > \frac{2\eta_1}{(1 + \theta)} + \frac{(2 + \mathcal{R}(\theta))\delta_Y + 3\mathcal{R}(\theta)\eta_2}{(1 - \mathcal{R}(\theta))(1 + \theta)}. \quad (3.38)$$

From relation (3.27) we get $\frac{\beta_E\eta_1\nu\nu_E}{\delta_F(\nu_E + \delta_E)} < \Delta\eta + (1 + \theta)(\eta_2 + \delta_Y)$. Thus

$$\begin{aligned} \phi &> \frac{2\eta_1}{(1 + \theta)} + \frac{(2 + \mathcal{R}(\theta))\delta_Y + 3\mathcal{R}(\theta)\eta_2}{(1 - \mathcal{R}(\theta))(1 + \theta)} - \frac{\Delta\eta\mathcal{R}(\theta) + (1 + \theta)(\eta_2 + \delta_Y)\mathcal{R}(\theta)}{(1 + \theta)^2}, \\ &> \frac{2\eta_1\mathcal{R}(\theta)}{(1 + \theta)} + \frac{(2 + \mathcal{R}(\theta))\delta_Y + 3\mathcal{R}(\theta)\eta_2}{(1 - \mathcal{R}(\theta))(1 + \theta)} + \frac{\eta_2\mathcal{R}(\theta)}{(1 + \theta)^2} - \frac{\eta_1\mathcal{R}(\theta)}{(1 + \theta)^2} - \frac{(\eta_2 + \delta_Y)\mathcal{R}(\theta)}{1 + \theta}, \\ &> \frac{\eta_1\mathcal{R}(\theta)(1 + 2\theta)}{(1 + \theta)^2} + \frac{2\mathcal{R}(\theta)\eta_2 + 2\delta_Y + \mathcal{R}(\theta)^2(\eta_2 + \delta_Y)}{(1 - \mathcal{R}(\theta))(1 + \theta)} + \frac{\eta_2\mathcal{R}(\theta)}{(1 + \theta)^2}, \\ &> 0. \end{aligned}$$

□

Proof of Theorem 3.3.1. Let $\alpha > 0$ and define $W : \mathcal{N} \rightarrow \mathbb{R}$ by

$$W((x^T, M_s)^T) := V(x) + \alpha \frac{(\theta M - M_s)^2}{\theta M + M_s} \text{ if } M + M_s \neq 0, \quad (3.39)$$

$$W((x^T, M_s)^T) := V(x) \text{ if } M + M_s = 0. \quad (3.40)$$

We have

$$W \text{ is continuous,} \quad (3.41)$$

$$W \text{ is of class } \mathcal{C}^1 \text{ on } \mathcal{N} \setminus \left\{(E, M, Y, F, U, M_s)^T \in \mathcal{N}; M + M_s = 0\right\}, \quad (3.42)$$

$$W((x^T, M_s)^T) \rightarrow +\infty \text{ as } \|x\| + M_s \rightarrow +\infty, \text{ with } x \in \mathcal{D} \text{ and } M_s \in [0, +\infty), \quad (3.43)$$

$$W((x^T, M_s)^T) > W(\mathbf{0}) = 0, \forall (x^T, M_s)^T \in \mathcal{N} \setminus \{\mathbf{0}\}. \quad (3.44)$$

From now on, and until the end of this proof, we assume that $(x^T, M_s)^T$ is in $\mathcal{N}$ and until (3.58) below we further

assume that

$$(M, M_s) \neq (0, 0). \quad (3.45)$$

One has

$$\begin{aligned} \dot{W}((x^T, M_s)^T) &= \nabla V(x)^T \cdot f(x, M_s) + \alpha(\theta M - M_s) \frac{2(\theta \dot{M} - \dot{M}_s)(\theta M + M_s) - (\theta \dot{M} + \dot{M}_s)(\theta M - M_s)}{(\theta M + M_s)^2}, \\ &= \nabla V(x)^T \cdot f(x, \theta M) + \nabla V(x)^T \cdot (f(x, M_s) - f(x, \theta M)) \\ &\quad + \alpha(\theta M - M_s) \frac{\theta \dot{M}(\theta M + 3M_s) - \dot{M}_s(3\theta M + M_s)}{(\theta M + M_s)^2}. \end{aligned}$$

Since

$$\begin{aligned} \nabla V(x)^T \cdot (f(x, M_s) - f(x, \theta M)) &= \begin{pmatrix} \frac{(1+2\mathcal{R}(\theta))\nu\nu_E}{(\nu_E+\delta_E)(1-\mathcal{R}(\theta))} \\ \nu \\ \frac{3\mathcal{R}(\theta)}{(1-\mathcal{R}(\theta))} \\ \frac{(2+\mathcal{R}(\theta))\beta_E\nu\nu_E}{\delta_F(\nu_E+\delta_E)(1-\mathcal{R}(\theta))} \\ \frac{\eta_1\beta_E\nu\nu_E\mathcal{R}(\theta)}{(1+\theta)\eta_2(\nu_E+\delta_E)\delta_F} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -\frac{\Delta\eta(\theta M - M_s)}{(M+M_s)(1+\theta)}Y \\ \frac{\eta_1(\theta M - M_s)}{(M+M_s)(1+\theta)}Y \\ -\frac{\eta_2(\theta M - M_s)}{(M+M_s)(1+\theta)}Y \end{pmatrix} \\ &= \frac{\phi Y(\theta M - M_s)}{M + M_s}, \end{aligned}$$

$$\begin{aligned} \dot{W}((x^T, M_s)^T) &= \nabla V(x)^T \cdot f(x, \theta M) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \\ &\quad \left[\frac{(\nabla V(x) \cdot (f((x^T, M_s)^T) - f(x, \theta M)))(\theta M + M_s)^2}{\alpha(\theta M - M_s)} \right. \\ &\quad \left. + \theta \dot{M}(\theta M + 3M_s) - \dot{M}_s(3\theta M + M_s) \right] \\ &= \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + ((1-\nu)\nu_E\theta E - \theta\delta_M M)(\theta M + 3M_s) \right. \\ &\quad \left. - u(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s) \right]. \end{aligned} \quad (3.46)$$

We take u as given by (3.36). Therefore, in the case where

$$\begin{aligned} \frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + ((1-\nu)\nu_E\theta E - \theta\delta_M M)(\theta M + 3M_s) \\ + \delta_s M_s(3\theta M + M_s) + \frac{1}{\alpha}(\theta M - M_s)(3\theta M + M_s) > 0, \end{aligned} \quad (3.47)$$

$$\begin{aligned} u &= \frac{1}{3\theta M + M_s} \left[\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + ((1-\nu)\nu_E\theta E - \theta\delta_M M)(\theta M + 3M_s) \right. \\ &\quad \left. + \delta_s M_s(3\theta M + M_s) + \frac{1}{\alpha}(\theta M - M_s)(3\theta M + M_s) \right], \end{aligned}$$

which, together with (3.46), leads to

$$\dot{W}((x^T, M_s)^T) = \dot{V}(x) - \frac{(\theta M - M_s)^2(3\theta M + M_s)}{(\theta M + M_s)^2}. \quad (3.48)$$

Otherwise, i.e. if (3.47) does not hold,

$$\begin{aligned} \frac{\phi Y(\theta M + M_s)^2}{\alpha(M + \gamma_s M_s)} + ((1-\nu)\nu_E\theta E - \theta\delta_M M)(\theta M + 3M_s) \\ + \delta_s M_s(3\theta M + M_s) + \frac{1}{\alpha}(\theta M - M_s)(3\theta M + M_s) \leq 0, \end{aligned} \quad (3.49)$$

so, by (3.36),

$$u = 0. \quad (3.50)$$

We consider two cases. If $\theta M > M_s$ using (3.46), (3.49) and (3.50)

$$\begin{aligned}\dot{W}((x^T, M_s)^T) &\leq \dot{V}(x) - \frac{(\theta M - M_s)^2(3\theta M + M_s)}{(\theta M + M_s)^2}, \\ &\leq -cV(x) - \frac{(\theta M - M_s)^2}{\theta M + M_s}, \\ &\leq -c_1 W((x^T, M_s)^T),\end{aligned}\tag{3.51}$$

with

$$c_1 := \min\{c, \frac{1}{\alpha}\} > 0.\tag{3.52}$$

Otherwise, if $\theta M \leq M_s$, using once more (3.46) and (3.50)

$$\begin{aligned}\dot{W}((x^T, M_s)^T) &= \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} \right. \\ &\quad \left. + \theta((1 - \nu)\nu_E E - \delta_M M)(\theta M + 3M_s) + \delta_s M_s(3\theta M + M_s) \right].\end{aligned}\tag{3.53}$$

$$\begin{aligned}\dot{W}((x^T, M_s)^T) &= \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + \theta((1 - \nu)\nu_E E) \right] \\ &\quad + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[-\delta_M M(\theta M + 3M_s) + \delta_s M_s(3\theta M + M_s) \right].\end{aligned}\tag{3.54}$$

From Lemma 3.3.1 we deduce that $\phi > 0$ and as $(x^T, M_s)^T \in \mathcal{N}$, one has $\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + \theta(1 - \nu)\nu_E E \geq 0$.

$$\text{So, } \theta M - M_s \leq 0 \implies \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + \theta(1 - \nu)\nu_E E \right] \leq 0.\tag{3.55}$$

The equation (3.54) becomes

$$\dot{W}((x^T, M_s)^T) \leq \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[-\theta \delta_M M(\theta M + 3M_s) + \delta_s M_s(3\theta M + M_s) \right].\tag{3.56}$$

The inequality (3.9) gives $\delta_s \geq \delta_M$ and one has

$$\begin{aligned}-\theta \delta_M M(\theta M + 3M_s) + \delta_s M_s(3\theta M + M_s) &\geq -\theta \delta_M M(\theta M + 3M_s) + \delta_M M_s(3\theta M + M_s) \\ &\geq \delta_M (M_s - \theta M)(M_s + \theta M).\end{aligned}\tag{3.57}$$

(3.57) together with $\theta M - M_s \leq 0$, implies that

$$\begin{aligned}\dot{W}((x^T, M_s)^T) &\leq \dot{V}(x) - \alpha \delta_M \frac{(\theta M - M_s)^2}{(\theta M + M_s)}, \\ &\leq -c_2 W((x^T, M_s)^T),\end{aligned}\tag{3.58}$$

with

$$c_2 := \min\{c, \delta_M\} > 0.\tag{3.59}$$

Let us now deal with the case where (3.45) is not satisfied. Note that, for every $\tau \geq 0$, $M(\tau) + M_s(\tau) > 0$ implies that $M(t) + M_s(t) > 0$ for all $t \geq \tau$. Thus, if $M(0) + M_s(0) = 0$, there exists $t_s \in [0, +\infty]$ such that $M(t) + M_s(t) = 0$ if and only if $t \in [0, t_s] \setminus \{+\infty\}$. Let us study only the case $t_s \in (0, +\infty)$ (the case $t_s = 0$ is obvious and the case $t_s = +\infty$ is a corollary of our study of the case $t_s \in (0, +\infty)$). Let us first point out that, for every $(M, M_s)^T \in [0, +\infty)^2$ such that $M + M_s > 0$, one has

$$\begin{aligned}\frac{M}{M + M_s} &\leq 1, \\ (\theta M + M_s)^2 &\leq (3\theta M + M_s)^2 \text{ and } \frac{(\theta M + M_s)^2}{(M + M_s)(3\theta M + M_s)} \leq \frac{(3\theta M + M_s)}{M + M_s} \leq 3\theta + 1, \\ \frac{\theta M + 3M_s}{3\theta M + M_s} &= \frac{\theta M}{3\theta M + M_s} + \frac{3M_s}{3\theta M + M_s} \leq \frac{1}{3} + 3 \leq 4.\end{aligned}$$

So

$$\frac{M}{M + M_s} \in [0, 1], \quad \frac{(\theta M + M_s)^2}{(M + M_s)(3\theta M + M_s)} \in [0, 3\theta + 1], \quad \text{and} \quad \frac{\theta M + 3M_s}{3\theta M + M_s} \in [0, 4]. \quad (3.60)$$

Let $t \mapsto \mathcal{X}(t) = (E(t), M(t), Y(t), F(t), U(t), M_s(t))^T$ be a solution (in the Filippov sense) of the closed-loop system (3.17)-(3.21) such that, for some $t_s \in (0, +\infty)$

$$M(t) + M_s(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.61)$$

Note that (3.61) implies that

$$M(t) = M_s(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.62)$$

From (3.60), (3.62) and the definition of a Filippov solution, one has on $(0, t_s)$

$$\begin{pmatrix} \dot{E} \\ \dot{M} \\ \dot{Y} \\ \dot{F} \\ \dot{U} \\ \dot{M}_s \end{pmatrix} = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E - \kappa(t)\Delta\eta Y - (\eta_2 + \delta_Y)Y \\ \eta_1 Y \kappa(t) - \delta_F F \\ \eta_2(1 - \kappa(t))Y - \delta_U U \\ Y g_1(t) + E g_2(t) - \delta_s M_s \end{pmatrix} \quad (3.63)$$

with

$$\kappa(t) \in [0, 1], \quad g_1(t) \in \frac{\phi}{\alpha}[0, 3\theta + 1] \quad \text{and} \quad g_2(t) \in (1 - \nu)\nu_E \theta[0, 4]. \quad (3.64)$$

From (3.62) and the second line of (3.63), one has

$$E(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.65)$$

From the first line of (3.63) and (3.65), we get

$$F(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.66)$$

Let us first consider the case where $Y(0) = 0$. Then, from the third line of (3.63) and (3.65), one has

$$Y(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.67)$$

To summarize, from (3.62), the fifth line of (3.63), (3.65), (3.66), and (3.67)

$$E(t) = M(t) = Y(t) = F(t) = M_s(t) = 0 \quad \text{and} \quad \dot{U}(t) = -\delta_U U(t), \quad \forall t \in [0, t_s], \quad (3.68)$$

which, with (3.28), (3.31), and (3.40), gives

$$\dot{W}(t) = -\sigma\delta_U U(t) \leq -\delta_U W(t), \quad \forall t \in [0, t_s]. \quad (3.69)$$

Let us finally consider the case where $Y(0) > 0$. Then, from the third line of (3.63),

$$Y(t) > 0, \quad \forall t \in [0, t_s], \quad (3.70)$$

which, together with the fourth line of (3.63) and (3.66), implies

$$\kappa(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.71)$$

To summarize, from (3.62), the third and the fifth line of (3.63), (3.65), (3.66), and (3.71),

$$E(t) = M(t) = F(t) = M_s(t) = 0, \quad \dot{Y}(t) = -(\eta_2 + \delta_Y)Y(t), \quad \text{and} \quad \dot{U}(t) = \eta_2 Y - \delta_U U(t), \quad \forall t \in [0, t_s],$$

which, with (3.28), (3.29), (3.31), and (3.40), gives

$$\begin{aligned} \dot{W}(t) &= -(\eta_2 + \delta_Y) \frac{3\mathcal{R}(\theta)}{(1 - \mathcal{R}(\theta))} Y(t) + \eta_2 \sigma Y(t) - \sigma\delta_U U(t) \\ &= -\mathcal{R}(\theta) \left((\eta_2 + \delta_Y) \frac{3}{(1 - \mathcal{R}(\theta))} - \frac{\eta_1 \beta_E \nu \nu_E}{(1 + \theta)(\nu_E + \delta_E)\delta_F} \right) Y(t) - \sigma\delta_U U(t) \\ &= -\mathcal{R}(\theta) \left(\frac{Q}{(1 - \mathcal{R}(\theta))(1 + \theta)(\nu_E + \delta_E)\delta_F} \right) Y(t) - \sigma\delta_U U(t) \end{aligned} \quad (3.72)$$

where

$$Q := 3(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E)\delta_F - (1 - \mathcal{R}(\theta))\eta_1\beta_E\nu\nu_E. \quad (3.73)$$

To end the proof we have to prove that $Q > 0$. Using the relation (3.26) and (3.27) we have

$$\beta_E\eta_1\nu\nu_E < \mathcal{R}(\theta)\delta_F(\nu_E + \delta_E)\Delta\eta + \delta_F(\nu_E + \delta_E)(1 + \theta)(\eta_2 + \delta_Y). \quad (3.74)$$

Recall that $\Delta\eta = \eta_1 - \eta_2$. One has

$$\begin{aligned} Q &= 3(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E)\delta_F - \eta_1\beta_E\nu\nu_E + \mathcal{R}(\theta)\eta_1\beta_E\nu\nu_E \\ &> 2(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E)\delta_F - \mathcal{R}(\theta)\Delta\eta(\nu_E + \delta_E)\delta_F + \mathcal{R}(\theta)\eta_1\beta_E\nu\nu_E \\ &> 2(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E)\delta_F - \mathcal{R}(\theta)\eta_1(\nu_E + \delta_E)\delta_F + \mathcal{R}(\theta)\eta_1\beta_E\nu\nu_E + \mathcal{R}(\theta)\eta_2(\nu_E + \delta_E)\delta_F. \end{aligned}$$

From the relation (3.15), $\eta_1\beta_E\nu\nu_E > \delta_F(\nu_E + \delta_E)(\eta_1 + \delta_Y)$.

$$\begin{aligned} Q &> 2(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E)\delta_F - \mathcal{R}(\theta)\eta_1(\nu_E + \delta_E)\delta_F + \mathcal{R}(\theta)\delta_F(\nu_E + \delta_E)(\eta_1 + \delta_Y) + \mathcal{R}(\theta)\eta_2(\nu_E + \delta_E)\delta_F \\ &> 2(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E)\delta_F + \mathcal{R}(\theta)(\eta_2 + \delta_Y)(\nu_E + \delta_E)\delta_F \\ &> 0. \end{aligned}$$

We get

$$\dot{W}(t) \leq -c'W(t), \quad \forall t \in [0, t_s], \quad (3.75)$$

where

$$c' := \min\left\{\frac{Q}{3(1 + \theta)\delta_F(\nu_E + \delta_E)}, \delta_U\right\}. \quad (3.76)$$

This proves Theorem 3.3.1 and gives the global exponential stability. From (3.52), (3.59) and (3.76) we obtain an estimate on the exponential decay rate

$$c_p := \min\left\{c, \frac{1}{\alpha}, \delta_M, \frac{Q}{3(1 + \theta)\delta_F(\nu_E + \delta_E)}, \delta_U\right\}. \quad (3.77)$$

□

3.3.1 Numerical simulations

Note that η_1 represents the natural fertility rate in the mosquito population. Wild males have a shorter maturity time in their life cycle than females. Thus, the fertilization phase is essentially around the hatching site. Sterile males are artificially released into the intervention region. We denote by p with $(0 \leq p \leq 1)$ the proportion of sterile males that are released. Also, the effective fertilization during the mating could be diminished due to the sterilization, which leads us to assume that the effective mating rate of sterile insects is given by $q\eta_1$ with $0 \leq q \leq 1$. Putting together these assumptions we get that the probability for a young female to mate with sterile males is $\frac{\eta_2 M}{M + M_s}$ with $\eta_2 = pq\eta_1$. For the numerical simulation we take $\eta_1 = 1$ and $\eta_2 = 0.7$. The numerical simulations of the dynamics when applying the feedback (3.36) is given in figure 3.1. The parameters we use are given in the following table. With the parameters given in table 3.1, condition (3.27) is $\theta > 102,06$. We fix

Parameters	Description	Value
β_E	Effective fecundity	10
ν_E	Hatching parameter	0.05
δ_E	Mosquitoes in aquatic phase death rate	0.03
δ_F	Fertilized female death rate	0.04
δ_Y	Young female death rate	0.04
δ_M	Male death rate	0.1
δ_s	Sterilized male death rate	0.12
ν	Probability of emergence	0.49

Table 3.1: Value for the parameters of system (3.17)-(3.20) (see [18][102]). Units are days⁻¹ except for ν .

$K = 21000$ and we consider the persistence equilibrium $z_0 = (E^0, M^0, Y^0, F^0, U^0, M_s^0)$ as initial condition. That gives $E^0 = 20700, M^0 = 5300, Y_0 = 1500, F^0 = 13000$ and $U^0 = M_s^0 = 0$. We take $\theta = 290$ and $\alpha = 90$.

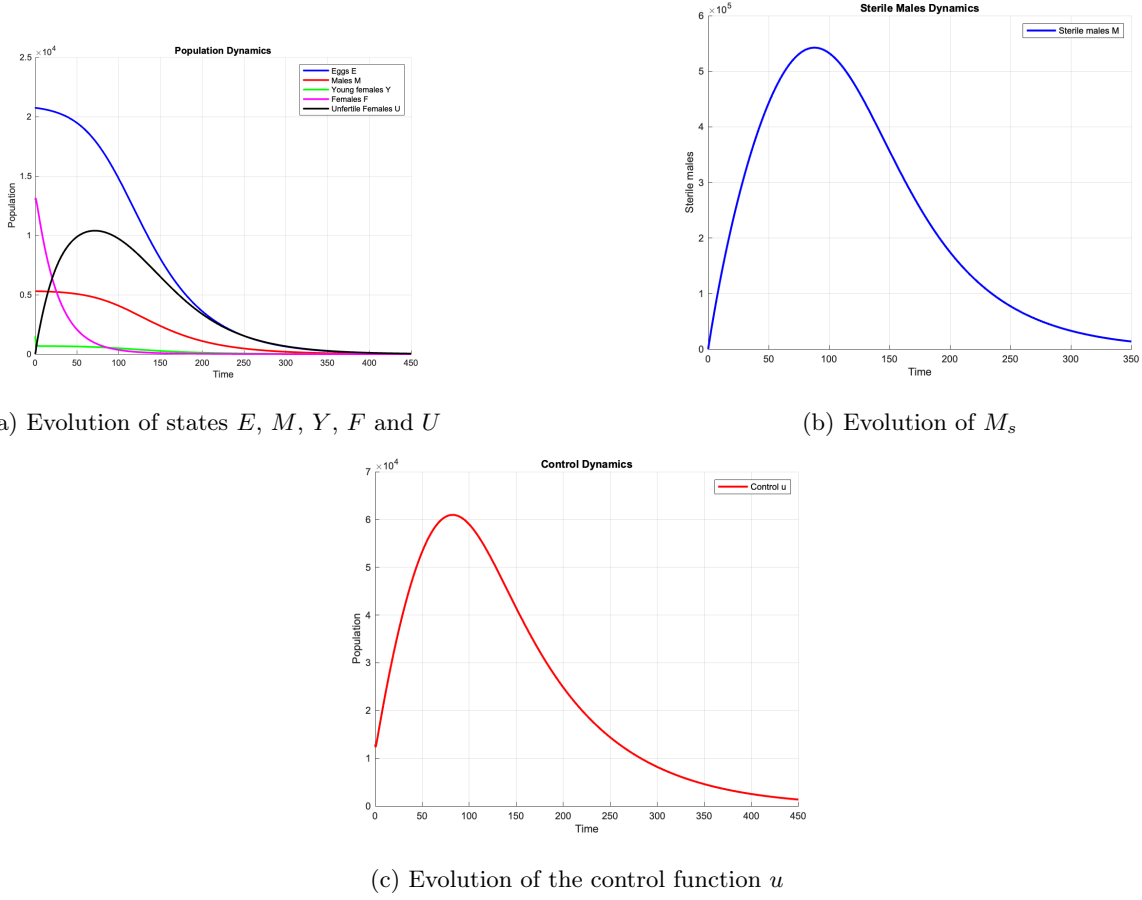


Figure 3.1: (a): Plot of E, M, Y, F and U when applying the feedback (3.36) with the initial condition z_0 . (b): Plot of M_s . (c): Plot of the feedback control function u .

Remark 3.3.2. Note that the feedback satisfies

$$\sup_{\varepsilon \rightarrow 0} \{ |u(\mathcal{X})| : \mathcal{X} \in \mathcal{N}, \|X\|_1 \in \mathcal{B}(0, \varepsilon) \} \rightarrow 0. \quad (3.78)$$

The advantage of applying feedback control is that when the density of the target population decreases, the control also decreases.

Remark 3.3.3. It is important to note that the backstepping feedback control (3.36) does not depend on the environmental capacity K , which is also an interesting feature for the field applications. In the case $K = +\infty$, the equation (3.10) becomes

$$\dot{E} = \beta_E F - (\delta_E + \nu_E) E, \quad (3.79)$$

and we prove by the same process that the same feedback law (3.36), ensures the exponential stability of the SIT system (3.79), (3.18)-(3.22) with the same lower bound of the exponential convergence rate.

Our stabilization result is the following one.

Theorem 3.3.2. Assume that (3.27) holds and $K = +\infty$. Then $\mathbf{0} \in \mathcal{N}$ is globally exponentially stable in $\mathcal{N}$ for system (3.79), (3.18)-(3.22) with the feedback law (3.36). The exponential convergence rate is bounded by $c_p > 0$ defined in (3.77).

Remark 3.3.4. Let us assume that the heterogeneity of the intervention zone strongly impacts the mating of female mosquitoes with sterile males more than we would have estimated. Suppose the estimated mating rate

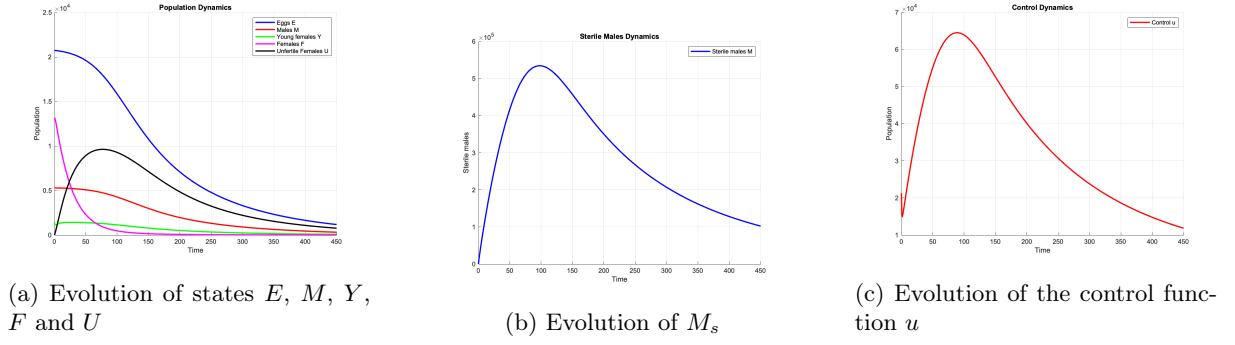


Figure 3.2: (a): Plot of E, M, Y, F and U when applying the feedback (3.36) with $\eta_2^e = 0.7$ while $\eta_2^r = 0.4$ for the dynamics. (b): Plot of M_s . (c): Plot of the feedback control function u .

for the control (3.36) is $\eta_2^e = 0.7$ and let the mating rate is $\eta_2^r = 0.4$ for the dynamics. Keeping the other parameters and the same initial condition, we obtain the following figure. This parameter considerably impacts the convergence time of the states of the system. Note that with $e = 3 \times 10^{-1}$ of error difference, we still have convergence. Estimation errors of the order of $e = 10^{-2}$ will have a negligible impact on the convergence time. This is because the backstepping control also depends on the states of the system. Thus, the states make a correction that can compensate for a certain margin of error. Unlike control, which only depends on the parameters, estimation errors have no correction from the dynamics. Therefore, this can be fatal to the success of the intervention. In practice, many external factors impact the life cycle of mosquitoes. These factors modify parameters such as birth, hatching, and fertilization rates. These factors are, for example, rainfall and the topography of the region. A SIT model that can integrate these factors is challenging to study (see [28]). The success of a SIT intervention depends strongly on the robustness of the control strategy. The results of our previous test that is reported in Figure 3.2, show us the advantage feedback control can provide in terms of robustness.

3.4 Observer design for SIT model

The application of feedback control requires measuring states such as eggs E and young females Y of the intervention zone over time. In practice, it is always important to estimate the density of adult mosquitoes to intervene in an area. These data are collected using mosquito traps distributed throughout the region. Despite various technological advances to improve these traps, it should be noted that some data are still easier to be measured than others. Measuring mosquito density in the aquatic phase E is difficult, specially in a heterogeneous area. It is also challenging to measure young females Y because females come in three categories, and we need to distinguish between unfertilized and fertilized females. Males are more easily measured because they are distinguishable. It can be also easy to distinguish wild males from laboratory males by marking processes applied to laboratory males. In this part of our paper, we will assume that the density of wild males and that of sterile males can be measured continuously. Our objective is to estimate the other densities. Observer design for nonlinear dynamic systems is a technique used in control theory to estimate the states of a system when only partial or indirect measurements are available. The difficulties in dealing with observer problems for general nonlinear systems is the proof of global convergence of the estimation error. Much literature exists on state observers and filters for nonlinear systems as they play crucial roles in control theory. To simplify the nonlinearity $F(1 - \frac{E}{K})$ of the SIT model, in this section we consider the simplified SIT model for environmental capacity $K = +\infty$. On the one hand, the reason for studying such a model is that the simplified model can be considered relevant from a biological point of view within a large intervention domain or in areas where environmental capacity is difficult to estimate. On the other hand, based on the result presented in Theorem 3.3.2, the proposed feedback law (3.36) still stabilizes the simplified model around zero with the same convergence rate. We consider the following output

control system:

$$\dot{E} = \beta_E F - (\delta_E + \nu_E)E, \quad (3.80)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (3.81)$$

$$\dot{Y} = \nu\nu_E E - \frac{\Delta\eta M}{M + M_s} Y - (\eta_1 + \delta_Y)Y, \quad (3.82)$$

$$\dot{F} = \frac{\eta_1 M}{M + M_s} Y - \delta_F F, \quad (3.83)$$

$$\dot{U} = \frac{\eta_2 M_s}{M + M_s} Y - \delta_U U, \quad (3.84)$$

$$\dot{M}_s = u - \mu_s M_s, \quad (3.85)$$

$$y_1 = M, \quad (3.86)$$

$$y_2 = M_s, \quad (3.87)$$

where the states is $X = (E, M, Y, F, U, M_s)^T \in \mathcal{N}$, the control is $u \in [0, +\infty)$ and the output is $y = (M, M_s)^T \in \mathbb{R}_+^2$.

In particular, in this model we are confronted with a difficulty in which most observer construction theories are invalid because of the singularity at the origin. To go around this difficulty, we will use the fact that the main nonlinearity term $\frac{M}{M+M_s}$ is bounded and essentially the most accessible data to measure. This leads us to develop an observer for this type of system.

3.4.1 Observer design for a class of nonlinear systems

The usual observers for linear systems are the Luenberger observer and the Kalman observer. Observer design for a nonlinear system is a complex problem in control theory and has received much attention from many authors yielding a large literature of methods. Among them, the most famous are the change of coordinates to transform the nonlinear system into a linear system [65, 20, 71, 25, 35] and a second approach consists in using the Extended Kalman Filter (EKF) [36, 90, 70, 108]. The state observer is called an exponential state observer if the observer error converges exponentially to zero. In this section we provide an explicit construction of a global observer for the following system.

$$\begin{cases} \dot{x}(t) = Ax(t) + B(y(t))x(t) + Du(t), \\ y(t) = Cx(t), \end{cases} \quad (3.88)$$

where $x(t) \in \mathbb{R}^n$, is the state vector, $u(t) \in \mathbb{R}^p$ is the input vector and $y(t) \in \mathbb{R}^m$ is the output vector. $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$ are the appropriate matrices. The matrice $B(y(t))$ is in the form

$$B(y(t)) = \sum_{i,j=1}^{n,n} b_{ij}(y(t))e_n(i)e_n^T(j). \quad (3.89)$$

We assume that for all $y(t) \in \mathbb{R}^m$ the coefficients b_{ij} are bounded for all $i = 1, \dots, n$ and $j = 1, \dots, n$ and denote

$$\bar{b}_{ij} = \max_t(b_{ij}(y(t))) \text{ and } \underline{b}_{ij} = \min_t(b_{ij}(y(t))). \quad (3.90)$$

Then, the parameter vector $b(t)$ remains in a bounded convex domain $\mathcal{S}_{n,n}$ of which $2^{(n^2)}$ vertices are defined by:

$$\mathcal{V}_{\mathcal{S}_{n,n}} = \{\eta = (\eta_{11}, \dots, \eta_{1n}, \dots, \eta_{nn}) | \eta_{ij} \in \{\underline{b}_{ij}, \bar{b}_{ij}\}\}.$$

A state observer corresponding to (3.88) is given as follows:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + B(y(t))\hat{x}(t) + Du(t) - L(C\hat{x}(t) - y(t)), \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \quad (3.91)$$

where $\hat{x}(t)$ denotes the estimate of the state $x(t)$. The dynamics of the observer error $e(t) := \hat{x}(t) - x(t)$ is $\dot{e}(t) = (A - LC)e(t) + B(y(t))e(t) = (A + B(y(t)) - LC)e(t)$. We define

$$\mathcal{A}(b(t)) = A + \sum_{i,j=1}^{n,n} b_{ij}(y(t))e_q(i)e_n^T(j). \quad (3.92)$$

The dynamics of the observer error becomes

$$\dot{e}(t) = (\mathcal{A}(b(t)) - LC)e(t). \quad (3.93)$$

The observation problem consists in finding a gain L such that (3.93) converges exponentially towards zero. We use the following results in [115].

Theorem 3.4.1. *The observer error converges exponentially towards zero if there exist matrices $P = P^T > 0$ and R of appropriate dimensions such that following Linear Matrix Inequalities (LMIs) are feasible:*

$$\mathcal{A}^T(\eta)P - C^T R + P\mathcal{A}(\eta) - R^T C + \xi I < 0, \quad (3.94)$$

$$\forall \eta \in \mathcal{V}_{\mathcal{S}_{n,n}}, \quad (3.95)$$

for some constant $\xi > 0$. When these LMIs are feasible, the observer gain L is given by $L = P^{-1}R^T$.

Proof. We follow [115] and consider the following quadratic Lyapunov function

$$\mathcal{V}(e) = e^T P e, \quad (3.96)$$

where P is the matrix in Theorem 3.4.1. We have $\dot{\mathcal{V}}(e)(t) = e(t)^T F(b(t))e(t)$, where $F(b(t)) = (\mathcal{A}(b(t)) - LC)^T P + P(\mathcal{A}(b(t)) - LC)$. For $e(t) \neq 0$ the condition $\mathcal{V}(e(t)) > 0$ is satisfied because $P > 0$ and the condition $\dot{\mathcal{V}}(e(t)) < 0$ is satisfied if we have

$$F(b(t)) < 0 \text{ for all } b(t) \in \mathcal{S}_{n,n}. \quad (3.97)$$

Since the matrix function F is affine in $b(t)$, using a convexity argument we deduce that $\forall t \geq 0$

$$\dot{\mathcal{V}}(e(t)) < -\xi \|e(t)\|_P^2, \quad (3.98)$$

if the following condition is satisfied $F(\eta) < -\xi I$, $\forall \eta \in \mathcal{V}_{n,n}$. Thus, if (3.94) holds, this inequality is also satisfied. $\square$

3.4.2 Application to the SIT model

We rewrite the output SIT models (3.80) -(3.86) as

$$\begin{cases} \dot{X} = AX + B(y)X + Du, \\ y = CX, \end{cases} \quad (3.99)$$

where $X = (E, M, Y, F, U, M_s)^T$,

$$A = \begin{pmatrix} -(\delta_E + \nu_E) & 0 & 0 & 0 & \beta_E & 0 \\ (1 - \nu)\nu_E & -\delta_M & 0 & 0 & 0 & 0 \\ \nu\nu_E & 0 & -(\eta_2 + \delta_Y) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_F & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_U & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta_s \end{pmatrix}, \quad B(y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Delta\eta \frac{y_1}{y_1 + y_2} & 0 & 0 & 0 \\ 0 & 0 & \eta_1 \frac{y_1}{y_1 + y_2} & 0 & 0 & 0 \\ 0 & 0 & \eta_2 \frac{y_2}{y_1 + y_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad D = (0, 0, 0, 0, 0, 1)^T.$$

As, $\mathcal{N}$ is an invariant set, one has $0 \leq \frac{y_1}{y_1 + y_2} \leq 1$. Solving the corresponding equation of (3.94) with $\xi = 1$ in MATLAB, we get

$$P = 10^4 \begin{pmatrix} 0.0219 & -0.1567 & -0.1531 & -0.1703 & -0.0344 & 0 \\ -0.1567 & 8.9301 & -0.8472 & -0.8081 & -0.4929 & 0 \\ -0.1531 & -0.8472 & 4.5716 & 0.9277 & 1.0845 & 0 \\ -0.1703 & -0.8081 & 0.9277 & 4.3088 & -2.3012 & 0 \\ -0.0344 & -0.4929 & 1.0845 & -2.3012 & 4.7413 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.7267 \end{pmatrix}, \quad (3.100)$$

$$R = 10^3 \begin{pmatrix} 0.2352 & 0.9704 & -0.4415 & -1.1401 & 0.0690 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.4162 \end{pmatrix}, \quad (3.101)$$

$$L = \begin{pmatrix} 50.6342 & 0 \\ 1.4150 & 0 \\ 0.9426 & 0 \\ 2.6547 & 0 \\ 1.6023 & 0 \\ 0 & 0.3800 \end{pmatrix}. \quad (3.102)$$

With the parameters given in table 3.1, the result for a simulation run with $x_0 = (400, 100, 150, 120, 120, 50)^T$,

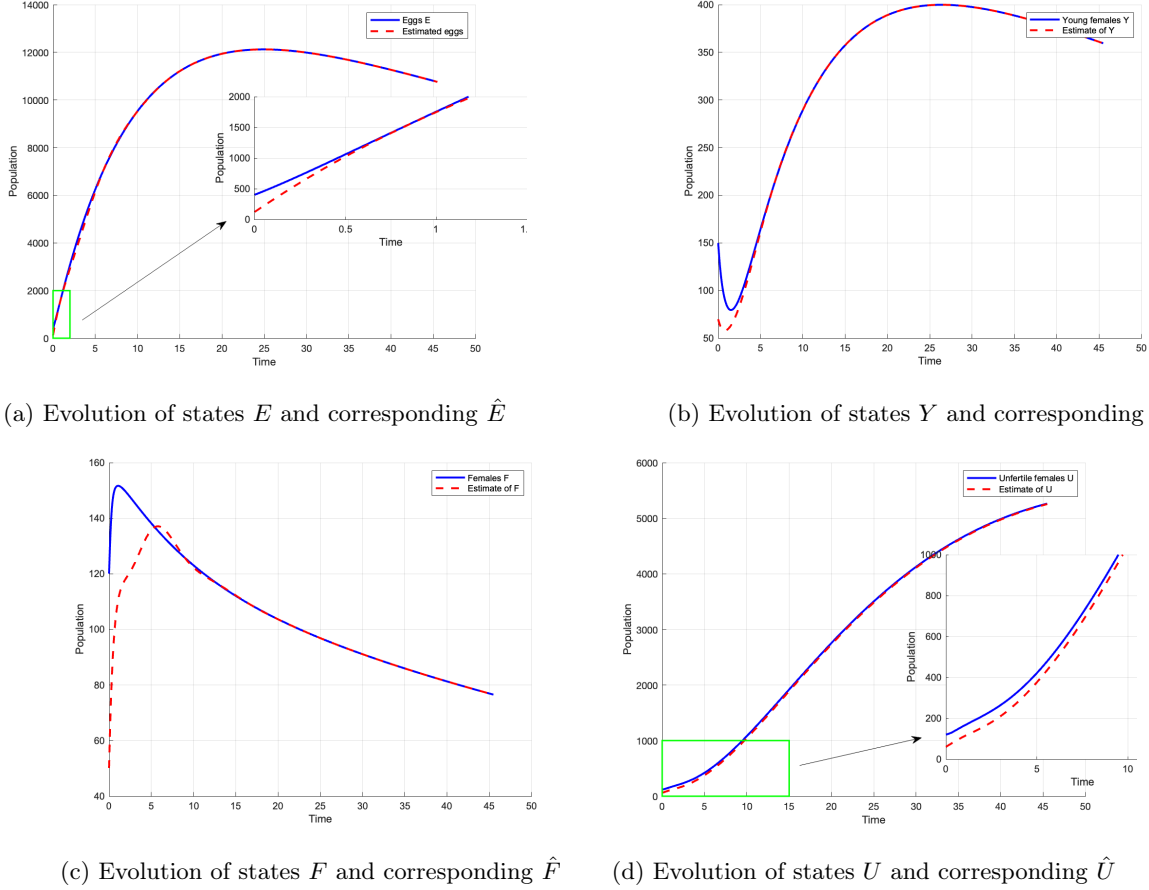


Figure 3.3: Simulation of the system and the observer under the constant input. The dashed lines illustrate the state estimate curves

$\hat{x}_0 = (120, 70, 70, 50, 60, 0)^T$ and $u = 500000$ is plotted in Figure 3.3. The asymptotic behavior of the different estimates $\hat{E}, \hat{F}, \hat{Y}$ and $\hat{U}$ (dashed), illustrates the exponential convergence of the estimation error show in Theorem 3.4.1.

3.5 Dynamic output feedback

The feedback control (3.36) depends on the states E , M , Y and M_s . From the measurement of states M and M_s , an observer system has been built in the previous section. This state observer is used to estimate both eggs E and young females Y . In this section we show that $\mathbf{u}(\hat{X}, y)$ stabilizes the dynamics at the origin. We consider the coupled system

$$\begin{cases} \dot{X} = f(X, \hat{\mathbf{u}}(\hat{X}, y)), \\ \dot{\hat{X}} = f(\hat{X}, \hat{\mathbf{u}}(\hat{X}, y)) - L(C\hat{x} - y), \end{cases} \quad (3.103)$$

with

$$\hat{\mathbf{u}}(\hat{X}, y) = \max(0, S(\hat{X}, y)). \quad (3.104)$$

where $S : \mathbb{R}^4 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$, $(\hat{X}, y)^T \mapsto S(\hat{X}, y)$ is defined by

$$S(\hat{X}, y) := G(\hat{E}, M, \hat{Y}, M_s) \quad (3.105)$$

The main result of this section is the following theorem.

Theorem 3.5.1. *Assume that (3.27) holds. Then $\mathbf{0} \in \mathcal{E} = \mathcal{N} \times \mathbb{R}^6$ is globally exponentially stable in $\mathcal{E}$ for system (3.103) with the feedback law (3.104). The convergence rate is bounded by the positive constant c_e defined by*

$$c_e := \min\{c_1, c_2, c', \frac{\xi}{4}\}. \quad (3.106)$$

Proof. Let $\lambda > 0$ and we define $H : \mathcal{E} \rightarrow \mathbb{R}$ by

$$H(X, \hat{X}) = W(X) + \lambda \sqrt{\mathcal{V}(e)} \quad (3.107)$$

with $e = \hat{X} - X$.

$$H \text{ is continuous on } \mathcal{E} \text{ and } C^1 \text{ on } \mathcal{E} \setminus \{(X, \hat{X}) \in \mathcal{E}; M + M_s = 0\}, \quad (3.108)$$

$$H(X, \hat{X}) \rightarrow +\infty \text{ as } \|(X, \hat{X})\| \rightarrow +\infty, \quad (3.109)$$

$$H(X, \hat{X}) > H(\mathbf{0}) = 0, \forall (X, \hat{X}) \in \mathcal{E} \setminus \{\mathbf{0}\}. \quad (3.110)$$

In this proof, from now on we assume that $(X, \hat{X})^T$ is in $\mathcal{E}$. Until (3.118) included, we also assume that

$$(M, M_s) \neq (0, 0). \quad (3.111)$$

One has

$$\begin{aligned} \dot{H}(X, \hat{X}) &= \dot{W}(X) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\phi Y (\theta M + M_s)^2}{\alpha (M + M_s)} \right. \\ &\quad \left. + ((1 - \nu)\nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \right. \\ &\quad \left. - \hat{\mathbf{u}}(\hat{X}, y)(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s) \right] + \lambda \frac{\dot{\mathcal{V}}(e)}{2\sqrt{\mathcal{V}(e)}}. \end{aligned}$$

Replacing the term $\hat{\mathbf{u}}(\hat{X}, y)$ by $\hat{\mathbf{u}}(\hat{X}, y) - \hat{\mathbf{u}}(X, y) + \hat{\mathbf{u}}(X, y)$ we get

$$\dot{H}(X, \hat{X}) = \dot{W}(X) + \alpha \frac{(\theta M - M_s)(3\theta M + M_s)}{(\theta M + M_s)^2} (\mathbf{u}(\hat{X}, y) - \mathbf{u}(X, y)) + \lambda \frac{\dot{\mathcal{V}}(e)}{2\sqrt{\mathcal{V}(e)}}. \quad (3.112)$$

Lemma 3.5.1. *There exist $C > 0$ such that, for all $(X, \hat{X}) \in \mathcal{E}$ and for all $y \in \mathbb{R}_+^2$,*

$$\|\hat{\mathbf{u}}(\hat{X}, y) - \hat{\mathbf{u}}(X, y)\| \leq C \|\hat{X} - X\|. \quad (3.113)$$

Note that $\dot{\mathcal{V}}(e) \leq -\xi \|e\|_P^2$. Thanks to this lemma, there exists $C' > 0$ independent of y such that

$$\dot{H}(X, \hat{X}) \leq \dot{W}(X) + C' \|e\| - \xi \lambda \frac{\|e\|_P^2}{2\sqrt{\mathcal{V}(e)}}. \quad (3.114)$$

Note that there exists a constant $\beta > 0$ such that and $\|e\| \leq \beta \|e\|_P$. So

$$\dot{H}(X, \hat{X}) \leq \dot{W}(X) - \left(\frac{\lambda \xi}{2} - \beta C'\right) \|e\|_P. \quad (3.115)$$

Hence for $\lambda = 4C'\beta/\xi$, and using the relation (3.52) and (3.59),

$$\dot{H}(X, \hat{X}) \leq -\min\{c_1, c_2\} W(X) - \frac{\lambda \xi}{4} \|e\|_P. \quad (3.116)$$

We conclude that there exists a constant

$$c_s := \min\{c_1, c_2, \frac{\xi}{4}\} \quad (3.117)$$

such that

$$\dot{H}(X, \hat{X}) < -c_s H(X, \hat{X}), \text{ if } M + M_s \neq 0. \quad (3.118)$$

Let us now deal with the case where (3.45) is not satisfied. As we explained previously in the proof of the Theorem 3.3.1, it is sufficient to study only the case $t_s \in (0, +\infty)$. Let $t \mapsto (E(t), M(t), Y(t), F(t), U(t), M_s(t), \hat{E}(t), \hat{M}(t), \hat{Y}(t), \hat{F}(t), \hat{U}(t), \hat{M}_s(t))$ be a solution (in the Filippov sense) of the closed-loop system (3.103) such that, for some $t_s \in (0, +\infty)$

$$M(t) + M_s(t) = 0 \quad \forall t \in [0, t_s] \quad (3.119)$$

Note that (3.119) implies that

$$M(t) = M_s(t) = 0, \quad \forall t \in [0, t_s] \quad (3.120)$$

From (3.60), (3.62) and the definition of a Filippov solution, one has on $(0, t_s)$

$$\begin{pmatrix} \dot{E} \\ \dot{M} \\ \dot{Y} \\ \dot{F} \\ \dot{U} \\ \dot{M}_s \end{pmatrix} = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E - \kappa(t)\Delta\eta Y - (\eta_2 + \delta_Y)Y \\ \eta_1 Y \kappa(t) - \delta_F F \\ \eta_2(1 - \kappa(t))Y - \delta_U U \\ \max(0, \hat{Y}g_1 + \hat{E}g_2) - \delta_s M_s \end{pmatrix} \quad (3.121)$$

$$\begin{pmatrix} \dot{\hat{E}} \\ \dot{\hat{M}} \\ \dot{\hat{Y}} \\ \dot{\hat{F}} \\ \dot{\hat{U}} \\ \dot{\hat{M}}_s \end{pmatrix} = \begin{pmatrix} \beta_E \hat{F} - (\nu_E + \delta_E)\hat{E} \\ (1 - \nu)\nu_E \hat{E} - \delta_M \hat{M} \\ \nu\nu_E \hat{E} - \kappa(t)\Delta\eta \hat{Y} - (\eta_2 + \delta_Y)\hat{Y} \\ \eta_1 \hat{Y} \kappa(t) - \delta_F \hat{F} \\ \eta_2(1 - \kappa(t))\hat{Y} - \delta_U \hat{U} \\ \max(0, \hat{Y}g_1 + \hat{E}g_2) - \delta_s \hat{M}_s \end{pmatrix} - LC\hat{X}, \quad (3.122)$$

with

$$\kappa(t) \in [0, 1], \quad g_1(t) \in \frac{\phi}{\alpha}[0, 3\theta + 1] \quad \text{and} \quad g_2(t) \in (1 - \nu)\nu_E \theta[0, 4]. \quad (3.123)$$

From (3.120) and the second line of (3.121), one has

$$E(t) = 0, \quad \forall t \in [0, t_s] \quad (3.124)$$

From the first line of (3.121) and (3.124), we get

$$F(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.125)$$

In the case where $Y(0) = 0$, from the third line of (3.121) and (3.124), one has

$$Y(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.126)$$

To summarize, from (3.120), the fifth line of (3.121), (3.124), (3.125), and (3.126)

$$E(t) = M(t) = Y(t) = F(t) = M_s(t) = 0 \quad \text{and} \quad \dot{U}(t) = -\delta_U U(t), \quad \forall t \in [0, t_s], \quad (3.127)$$

which, with (3.28), (3.31), and (3.40), gives

$$\dot{W}(t) = -\sigma\delta_U U(t) \leq -\delta_U W(t), \quad \forall t \in [0, t_s]. \quad (3.128)$$

In the case where $Y(0) > 0$, from the third line of (3.121),

$$Y(t) > 0, \quad \forall t \in [0, t_s], \quad (3.129)$$

which, together with the fourth line of (3.121) and (3.125), implies

$$\kappa(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.130)$$

Referring to this case already studied in the proof of Theorem 3.3.1 we get

$$\dot{W}(t) \leq -c'W(t), \quad \forall t \in [0, t_s]. \quad (3.131)$$

$$\kappa(t) \in [0, 1], \quad g_1(t) \in \frac{\phi}{\alpha}[0, 3\theta + 1] \quad \text{and} \quad g_2(t) \in (1 - \nu)\nu_E \theta[0, 4], \quad (3.132)$$

$$\dot{M}_s(t) = \max(0, \hat{Y}g_1 + \hat{E}g_2) - \delta_s M_s \quad (3.133)$$

Since $M_s(t) = 0 \forall t \in [0, t_s]$, $\max(0, \hat{Y}g_1 + \hat{E}g_2) = 0$. For all $\kappa(t) \in [0, 1]$, in these two cases, the dynamics of the observation error remains

$$\dot{e} = (A(\kappa(t)) - LC)e, \quad (3.134)$$

and one has

$$\dot{H}(X, \hat{X}) = -c'W(X) - \frac{\lambda\xi}{2}\|e\|_P. \quad (3.135)$$

We conclude that there exists a constant

$$c_w := \min\{c', \frac{\xi}{2}\}, \quad (3.136)$$

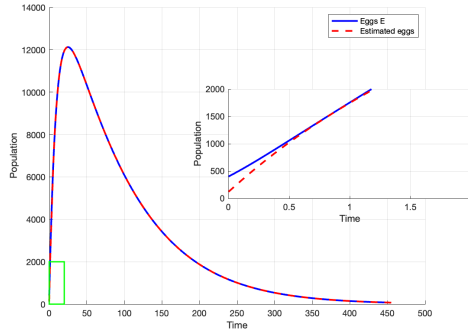
such that

$$\dot{H}(X, \hat{X}) \leq -c_w H(X, \hat{X}). \quad (3.137)$$

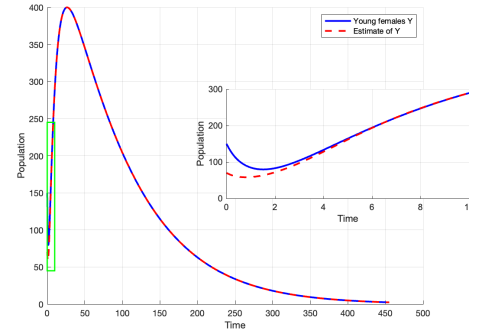
This proves Theorem 3.5.1 and gives the global exponential stability with the exponential decay rate c_e given by relation (3.106). $\square$

3.5.1 Numerical simulations

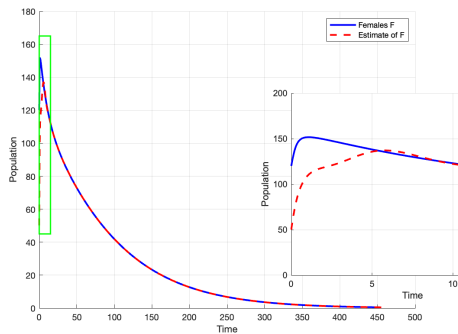
We apply the backstepping control u function of the measured states y and the estimated states $\hat{E}$ and $\hat{Y}$ given by the relation (3.104) with the following initial condition $x_0 = (20000, 5000, 1500, 12000, 500)$ and $\hat{x}_0 = (2000, 500, 150, 1200, 0)$. The response of system (3.103) to the backstepping control (3.104) is illustrated in



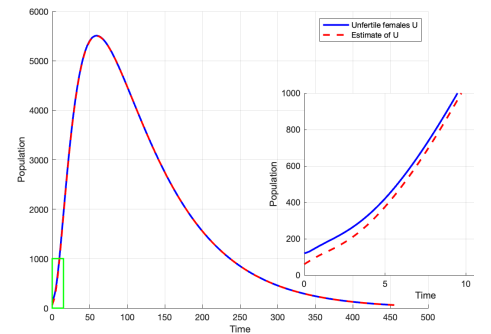
(a) Evolution of states E and estimate $\hat{E}$



(b) Evolution of states Y and estimate $\hat{Y}$



(c) Evolution of states F and estimate $\hat{F}$



(d) Evolution of states U and estimate $\hat{U}$

Figure 3.4: Simulation of the SIT model when applying backstepping feedback law with estimated and measured states (3.104) .

the Figure 3.4. The asymptotic convergence in large t of the different estimated $\hat{E}, \hat{F}, \hat{Y}$ and $\hat{U}$ (dashed) to their corresponding state variables E, F, Y and U respectively, illustrates the exponential convergence of the estimation error stated in Theorem 3.4.1. The convergence of the states and their estimates, towards zero, proves

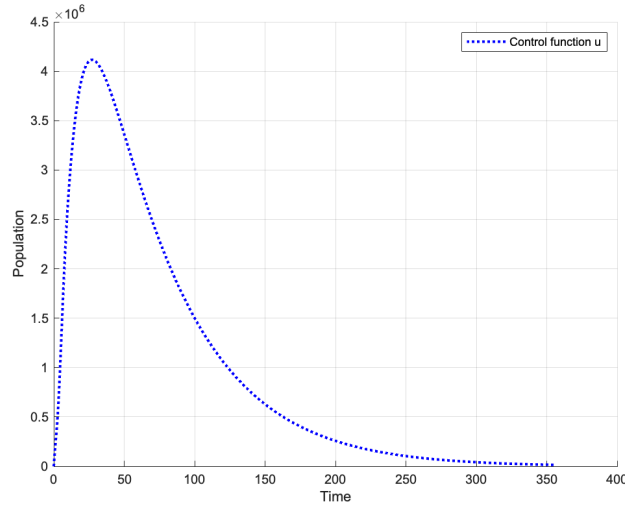


Figure 3.5: Evolution of control function $\mathbf{u}(\hat{X}, y)$.

the efficiency of the control(3.104) as was show in the Theorem 3.5.1. Figure 3.5 shows that the applied control function decreases, when the density of the target population decreases.

Conclusion

In this work, we have built a feedback control law to stabilize the SIT model presented in [50, 18] at extinction. Control by state feedback is a type of control rarely proposed in the literature for the overall stabilization of the SIT model. The feedback control (3.36) developed in this work has many advantages including robustness to changing parameters. We have shown in Remark 3.3.4 that despite the margin of error that can be made in the estimation of the parameters, this feedback control still makes the system converge to extinction. Moreover its does not depend on environmental capacity and this control law ensures exponential stability with the same convergence rate for the SIT system even in the high environmental capacity limit (see Theorem 3.3.2). Remark 3.3.2 shows that when the density of the target population decreases, the control also decreases

In section 3.4 of our work, we built an observer for the SIT model where, using the measurement of male mosquitoes, our state estimator gives us an estimate of the other states of the system. This aspect is rarely studied for this type of dynamics. An accurate estimate of the mosquito population enables resources to be allocated more efficiently. If the intervention is effective in some areas but not in others, resources can be reallocated to maximize impact. On the other hand, the data collected during the SIT intervention provides essential information on the impact of the control in the conditions of the intervention area. This will enable informed decisions on future control strategies to be adopted according to conditions in the intervention zone by adding complementary methods or by adapting existing approaches.

One of the applications we made was to show in section 3.5 that by using the data estimated via our observer to adjust the feedback control, we globally stabilize the system upon extinction. Figure 3.4 shows that the difficulty of estimating eggs and young females during an intervention can be compensated by the application of the observer system. Data collected on the mosquito population is also used in epidemic prevention programs. They help to adapt public health programs for better control of mosquito-borne diseases.

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Chapter 4

A Novel Approach to Feedback Control with Deep Reinforcement Learning

This chapter is a joint work with Hayat Amaury, Nathan Lichtlé and Jean-Michel Coron. Preliminary version published at The 3rd Workshop on Mathematical Reasoning and AI at NeurIPS'23 [4].

Abstract. We present a novel approach to feedback control design by leveraging the power of deep reinforcement learning (RL). The goal is to blend the RL methodology with mathematical analysis to extract an explicit feedback control for systems where, because of constraints or limited measurements, the classical approaches of control theory (e.g. dynamical programming, optimal control-based feedback, backstepping, etc.) cannot be used.

We study a dynamical system of mosquito populations for biological pest control using the Sterile Insect Technique (SIT), a method traditionally applied in agriculture that involves releasing large numbers of sterile insects into the wild to reduce pest populations. Our goal is to derive a feedback control that globally stabilizes the system around the zero-mosquito equilibrium using only practical measurements, such as total male and female mosquito counts, rather than detailed counts of sterilized versus potent males or fecund versus unfecund females, which are often not accessible. This physical constraint presents challenges for classical methods control theory, as the full state cannot be measured. To address this, we apply deep reinforcement learning to suggest feedback laws for a discretized system that only rely on these accessible, real-world measurements, obtainable through methods like pheromone traps. Finally, we leverage the trained neural network to extract explicit feedback controls that stabilize the original continuous system over a wide range of initial conditions.

Many other dynamical systems arising from practical applications are subject to measurement constraints, which render the stabilization problem complex from a mathematical perspective. We believe that this approach could help in finding new solutions to these problems.

4.1 Introduction

Stabilization is a branch of mathematics that is very useful for many applications. Many controls have been developed in the last century to address the local or global stabilization of dynamic systems [44]. One can cite for instance optimal control-based feedback, dynamical programming [26, 77], backstepping [37, 68, 106], and many others. However, in some cases, such methods cannot be used, for instance because the system suffers from physical constraints, such as limited measurements. Observers strategy to tackle this limitation have been developed but in many cases finding robust observers is still a challenge. In this paper, we introduce a new approach that employs deep reinforcement learning (RL) to suggest mathematical control strategies for dynamical systems. We illustrate it on a mathematical model of the Sterile Insect Technique (SIT) applied on mosquitoes population, namely (4.1)-(4.4) below. Controlling this system is particularly challenging because we only rely on infrequent, discrete measurements of the physically observable part of the state.

Our approach differs from classical methods and uses deep reinforcement learning (RL) to construct control feedback laws. Over the past few years, RL [103, 27] has emerged as a powerful approach for control, with its ability to learn near-optimal decision-making strategies through interactions with an environment, and has demonstrated remarkable successes across a wide range of domains and applications with long-term horizons, high-dimensional partially-observable states. In particular, RL can perform well on certain tasks exhibiting states that are too large, too complex to model or too partially-observable to be efficiently solved with traditional methods such as optimal control, the cost being that RL controllers typically do not inherently have safety guarantees,

explainability or optimality [27, 89]. In robotics, RL has enabled machines to learn complex tasks such as locomotion, manipulation, and dexterous object handling [76, 56, 80]. In the realm of games, RL algorithms have achieved superhuman performance in challenging domains like Go, Chess, StarCraft [97, 109], as well as in classical Atari games [82]. These remarkable achievements highlight the versatility and potential of RL as a general-purpose approach for solving complex practical control problems in diverse domains. Akin to robotics, RL has been used to solve systems of ODEs or PDEs [101], which can sometimes be augmented by enforcing constraints on the training of the RL algorithm [40] or directly on the neural networks, for instance with physically-informed neural networks (PINNs) [78]. [73] does RL through symbolic regression using a basis of functions, which has the advantage of always being explicit and thus more interpretable, but in general is likely less expressive than state-of-the-art RL algorithms coupled with neural networks. Other approaches to symbolic regression using RL and neural-guided genetic programming can be found in [83], although it considers learning explicit forms of simple mathematical formulas, which doesn't consider the mathematical system solving part.

RL techniques, while powerful for decision-making, inherently provide control mechanisms that are discrete and numerical in nature. However, from a more rigorous mathematical point of view, these mechanisms often don't translate directly into analytical feedback control formulas. Recognizing this limitation, in our work, we blend RL methodologies with mathematical analysis to extract an explicit mathematical control. Importantly, in our approach we apply RL to solve the mathematical problem, rather than solely applying RL to the discretized system, a more conventional application of RL. This allows us to employ deep RL to architect control feedback laws which can be juxtaposed with existing controls that have been derived from more traditional methodologies.

Our example of application is the Sterile Insect Technique (SIT): it is a method of biological pest control that consists of releasing sterilized insects to reduce or eliminate a target population. Initially used in agriculture to control insect pests, it is now employed in the vector-borne disease fight against mosquitoes that carry illnesses such as malaria and arboviruses [7, 15] and there is a great interest both in research and in practice to understand which control to use for releasing the sterilized insects [31, 30, 12].

Several mathematical approaches have already been used in the literature to treat the SIT control problem applied to mosquitoes either for the complete system or for reduced models. Two reduced models have been considered: a two dimensional (2D) model obtained by assuming that the dynamics of males and eggs are fast so that these two populations can be assumed to be at equilibrium (see [12, ($\mathcal{S}_1$), page 231-232] or [47, (2)]); and a three dimensional (3D) model obtained by overlooking the non-adult stages (see [32, (7a)-(7b)-(7c)]). These mathematical approaches have led to the following stabilizing feedback controls:

- Stabilization using impulsive feedback controls for the 3D model: [32, Theorem 6] and [32, Theorem 7] for the case of sparse measurements. The case of vector migration is also considered in [33].
- Stabilization using optimal feedback control of the reduced model 2D model and the complete model: [12] and in particular [12, Remark 3.4].
- Stabilization using the backstepping method: [47] considers the 2D model while [2, Section 3.1] considers the complete model. See, for example, [72, Section I.2.2], [99, Pages 242–246] or [44, Section 12.5] for tutorial presentations of the backstepping method.
- Stabilization using simple linear feedback laws: [2] proves stabilization for positively invariant subsets and conjectures it for the complete model.

These feedback controls above are constructed using classical tools in control theory such as control Lyapunov functions, the LaSalle invariance principle, the maximum principle, monotone dynamical systems, barrier functions, or the backstepping method. Although these feedback controls provide evidence in terms of robustness, their applications requires continuously measuring separately the different components of the state of the model, which is difficult or impossible in practice. Thank to our approach, we are able to provide a control only based on the most accessible data to be measured: the adult population (adult females and adult males) using pheromone traps, which are already used in practice.

4.2 Mathematical framework of the problem

The SIT model in mosquitoes population is given by the following system of equations:

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \quad (4.1)$$

$$\dot{M} = (1 - \nu) \nu_E E - \delta_M M, \quad (4.2)$$

$$\dot{F} = \nu \nu_E E \frac{M}{M + M_s} - \delta_F F, \quad (4.3)$$

$$\dot{M}_s = u - \delta_s M_s, \quad (4.4)$$

where, at time t , $E(t) \geq 0$ is the mosquito density in aquatic phase, $M(t) \geq 0$ is the wild adult male density, $F(t) \geq 0$ is the density of adult females which have been fertilized, $M_s(t) \geq 0$ is the sterilized adult male density, and $u(t) \geq 0$, the control, is the density of sterilized males released at time t .

In system (4.1)–(4.4) we assume that all females mate as soon as they emerge from the pupal stage. The density of unfertilized females, i.e. the density of females that have mated with sterilized males, is denoted by $F_s(t)$. One has $F_s(t) = F(t)M_s(t)/M(t)$. Besides, we also assume that $\delta_s \geq \delta_M$, which is usually considered to be a biologically relevant assumption [12]. The interpretation of the parameters are given below [12]:

- $\beta_E > 0$ is the oviposition rate,
- $\delta_E, \delta_M, \delta_F > 0$ are the death rates for eggs, wild adult males and fertilized females respectively,
- $\nu_E > 0$ is the hatching rate for eggs,
- $\nu \in (0, 1)$ the probability that a pupa gives rise to a female (and $(1 - \nu)$ is, therefore, the probability to give rise to a male),
- $\delta_s > 0$ is the death rate of sterilized adults,
- $K > 0$ is the environmental capacity for eggs. It can be interpreted as the maximum density of eggs that females can lay in breeding sites. Since here the larval and pupal compartments are not present, it can be interpreted as E representing all the aquatic compartments and this term K representing a logistic law's carrying capacity for the aquatic phase (that also includes the effects of competition between larvae).

Typical values for these parameters as well as the values used in this work are given in Table 4.5.

For the parameters given in Table 4.5, when $u(t) = M_s(t) = 0$ for any $t \geq 0$, the system (4.1)–(4.3) has a unique globally asymptotically stable equilibrium $(E(t), M(t), F(t)) \equiv (E^*, M^*, F^*)$ where E^* , M^* and F^* are large constant values. This corresponds to the situation where mosquitoes reproduce freely. The state $(E(t), M(t), F(t)) \equiv (0, 0, 0)$ is also an equilibrium, albeit an unstable one. The mathematical problem is to find $u(t)$ of the form

$$u(t) = f(M(t) + M_s(t), F(t) + F_s(t)), \quad (4.5)$$

where $f \in L^\infty(\mathbb{R}^2)$, such that the zero equilibrium $(0, 0, 0)$ is globally asymptotically stable and M_s is asymptotically small, meaning there exists $c \in \mathbb{R}_+$ such that

$$\lim_{t \rightarrow +\infty} \|u(t)\| = c < U^* \quad (4.6)$$

where

$$U^* := \frac{K\beta_E\nu(1-\nu)\nu_E^2\delta_s}{4(\delta_E + \nu_E)\delta_F\delta_M} \left(1 - \frac{\delta_F(\nu_E + \delta_E)}{\beta_E\nu\nu_E}\right)^2, \quad (4.7)$$

and the equilibrium $(0, 0, 0, c/\delta_s)$ of the system (4.1)–(4.4) is globally asymptotically stable (see Definition 4.2.1 below, where the notion of solutions of the closed-loop system is understood in the Fillipov sense [2, Section 2.2]). Ideally, one would even like to be able to find, for any $\varepsilon > 0$, a control feedback law f_ε such that

$$\lim_{t \rightarrow +\infty} u(t) = \varepsilon. \quad (4.8)$$

Definition 4.2.1. *The equilibrium $(0, 0, 0, c/\delta_s)$ of the system (4.1)–(4.4) with the feedback law (4.5) is globally asymptotically stable if, for any initial condition $(E_0, M_0, F_0, M_{s,0})$, the (forward maximal) solutions (E, M, F, M_s) to the system (4.1)–(4.4) with the feedback law (4.5) are defined on $[0, +\infty)$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|(E_0, M_0, F_0, M_{s,0} - c/\delta_s)\| \leq \delta \implies \|(E(t), M(t), F(t), M_s(t) - c/\delta_s)\| \leq \varepsilon, \quad \forall t \in [0, +\infty), \quad (4.9)$$

$$\lim_{t \rightarrow +\infty} \|(E(t), M(t), F(t), M_s(t) - c/\delta_s)\| = 0, \quad (4.10)$$

The form constraint (4.5) corresponds to a practical limitation due to the quantities that we are able to reliably measure in practice: $M + M_s$ and $F + F_s$ are the total number of males and females which are typically what can be measured in practice (see [7]), and in particular f cannot depend on E or M and M_s separately since these two quantities are difficult to measure. Nevertheless, we consider different variants in this work.

Remark 4.2.1 (Constant control). *The rationale behind the definition of (4.6) is that this is the critical value above which a constant control can stabilize the state $(E^*, M^*, F^*) = (0, 0, 0)$. Indeed, for a constant control $u(t) \equiv \bar{U}$, if $\bar{U} > U^*$ then the equilibrium $(E^*, M^*, F^*, M_s^*) = (0, 0, 0, U^*/\delta_s)$ is globally asymptotically stable (see [12]).*

Remark 4.2.2 (Optimal decay rate). Assume that $E(0) \leq K$. Then, for any control $u(t) \geq 0$ is, for every time $t \geq 0$, we have $E(t) \leq K$ and

$$E(t) \geq \tilde{E}(t), M(t) \geq \tilde{M}(t), F(t) \geq \tilde{F}(t), \quad (4.11)$$

where $(\tilde{E}, \tilde{M}, \tilde{F})$ is the solution to the Cauchy problem

$$\dot{\tilde{E}} = \beta_E \tilde{F} \left(1 - \frac{\tilde{E}}{K}\right) - (\nu_E + \delta_E) \tilde{E}, \quad (4.12)$$

$$\dot{\tilde{M}} = (1 - \nu) \nu_E \tilde{E} - \delta_M \tilde{M}, \quad (4.13)$$

$$\dot{\tilde{F}} = -\delta_F \tilde{F}, \quad (4.14)$$

$$(\tilde{E}(0), \tilde{M}(0), \tilde{F}(0)) = (E(0), M(0), F(0)). \quad (4.15)$$

It would be interesting to see if one can get with suitable output feedback laws (vanishing or small at the origin) a decay rate close to the one imposed by (4.11), i.e.

$$E(t) \simeq \tilde{E}(t), M(t) \simeq \tilde{M}(t), F(t) \simeq \tilde{F}(t). \quad (4.16)$$

(Note that it is possible to get (4.16) by taking u constant and large, depending on $(E(0), M(0), F(0))$.) This would be particularly useful in the case where the insect under study reproduces both sexually and asexually: indeed, this would give the best way to reduce the sexual reproduction part as much as possible by output feedback laws (vanishing or small at the origin).

In [2] a backstepping feedback control was built to stabilize this specific system at the origin. However, this control depends on the three variables (E, M, M_s) and not only on the feasibly observable quantities $M + M_s$ and $F + F_s$. As of now, there is no known control depending only on $M + M_s$ and $F + F_s$.

One possible approach could be to use the measurements of $M + M_s$ and $F + F_s$ to design an observer that reconstructs the rest of the state, enabling the application of backstepping control as an observer-based control. This has been successfully implemented when separate measurements on M and M_s are available [1]. However, it remains a challenge when only $M + M_s$ and $F + F_s$ are measured. The main difficulty arises from the term $M/(M + M_s)$, which is not globally Lipschitz and becomes singular at $(0, 0)$. This can pose a problem when $M + M_s$ is small and M is not directly measured. Consequently, certain common observer methods, such as extended Kalman filters for nonlinear systems and linearization techniques, are already inapplicable. To the best of our knowledge, no effective approach exists to address this challenge.

4.3 Approach and contributions of this paper

In this work we use a deep Reinforcement Learning (RL) approach to construct control feedback laws and compare them with the existing feedback controls that were deduced in previous work.

Our approach works in three steps:

1. We discretize the equations in a numerical scheme,
2. We train an RL model to obtain a numerical control feedback based on this numerical scheme,
3. We recover from the numerical control feedback an explicit mathematical control. We then perform several tests to ensure that the explicit control is efficient.

This is detailed in Section 4.4. We use this approach to construct three types of control feedback laws:

- A feedback control depending on M only,
- A feedback control depending on $M + M_s$ (total number of males) only,
- A feedback control depending on $M + M_s$ and $F + F_s$ (total number of males and of females).

These different types of controls are studied in Sections 4.5, 4.6 and 4.7 respectively.

4.4 Reinforcement learning based control

Reinforcement Learning (RL) is a subfield of machine learning concerned with training *agents* to make decisions in an environment to maximize a long-term *reward* function. From a control perspective this can equivalently be seen as finding an optimal control for a cost function over the trajectories. We first define these concepts more formally and discuss the RL algorithm we use in Section 4.4.1, then explain how we apply this formalism to our specific problem in Section 4.4.2. After that, we discuss our approach in Section 4.4.3 and finally go through experiment details in Section 4.4.4.

4.4.1 RL Background

We model the environment using the common formalism of a Partially-Observable Markov Decision Process (POMDP) [100] $\mathcal{M} = (\mathcal{S}, \mathcal{A}, T, R, \gamma, \mu, \Omega, \mathcal{O})$ where $\mathcal{S} \subseteq \mathbb{R}^n$ is a set of states, $\mathcal{A} \subseteq \mathbb{R}^m$ a set of actions, $T : \mathcal{S} \times \mathcal{A} \rightarrow \Delta\mathcal{S}$ is the state transition function (ie. $T(s'|s, a)$ is the probability of transitioning to state s' given state s and action a), $R : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ is the reward function, $\gamma \in [0, 1]$ is the discount factor, $\mu \in \Delta(\mathcal{S})$ is the initial state distribution, $\Omega \subseteq \mathbb{R}^p$ is a set of observations of the hidden state, and $\mathcal{O} : \mathcal{S} \rightarrow \Delta(\Omega)$ is the observation distribution (ie. $\mathcal{O}(o|s)$ is the probability of getting observation o given current state s). Note that given a set X , $\Delta(X)$ denotes the set of probability distributions over X .

The goal for the agent is to learn a policy $\pi_\theta : \Omega \rightarrow \Delta(\mathcal{A})$ (stochastic in our case) mapping observations to actions, where θ are the parameters of the policy (typically the weights of a neural network in the case of deep RL), which maximizes the expected discounted sum of rewards

$$J(\pi_\theta) = \mathbb{E}_{\tau \sim (\pi_\theta, \mathcal{M})} \left[\sum_{t=0}^{\infty} \gamma^t r_t \right] \quad (4.17)$$

where the expectation is taken over all trajectories $\tau = (s_t, a_t, r_t)_{t \geq 0}$ generated by the current policy π_θ acting in the POMDP $\mathcal{M}$. In other words, we aim to find optimal parameters $\theta^* = \arg \max_\theta J(\pi_\theta)$. Note that maximizing this objective $J(\pi_\theta)$ is analogous to the standard minimizing of the cost function $-J(\pi_\theta)$ in control theory.

One common way to minimize this cost is to use policy gradient methods, which directly optimize the policy parameters by estimating the gradient of the expected cumulative reward. The basic policy gradient algorithm updates the policy parameters in the direction of the estimated gradient to increase the likelihood of actions that lead to higher rewards:

$$\theta_{\text{new}} \leftarrow \theta_{\text{old}} + \nabla_{\theta_{\text{old}}} \hat{J}(\pi_{\theta_{\text{old}}}), \quad (4.18)$$

where $\hat{J}$ is the sample mean of J , calculated by averaging the discounted sum of rewards over a number of trajectories. To improve stability and convergence, several techniques have been developed, such as Trust Region Policy Optimization (TRPO) [94] and Proximal Policy Optimization (PPO) [95] which we use in this work. TRPO limits the policy update step size by constraining the divergence between the new policy and the old policy. PPO introduces a simple surrogate objective function that includes a clipping mechanism to prevent large policy updates and improve sample efficiency, while still maximizing the likelihood of high-reward actions, and vice versa. In parallel, PPO also learns a value function, which is a second neural network that aims to estimate the value of each state, i.e. the total amount of rewards that can be expected when starting from a given state. A second term in the loss function drives the learning of this value function, by minimizing its mean squared error with respect to the empirical values of the state obtained from policy trajectories. Finally, a third term encourages exploration by rewarding higher-entropy policies.

4.4.2 Defining the POMDP

In this section, we define the specific POMDP problem formulation that we consider in this work: state, observations, actions and rewards, as well as the initial state distribution and state transition function that model the system. We present all the variants that we have considered, and the specific settings used for each controller will be specified in the respective feedback law parts in Sections 4.5, 4.6 and 4.7.

State space The state for the POMDP are exactly the state of the SIT model, that is, $(E(t), M(t), F(t), M_s(t))^T$, all components being nonnegative and introduced in Section. 4.2. Thus our state space can be formally written as $\mathcal{S} = \mathbb{R}_{\geq 0}^4$.

Observation space and distribution To account for real-world partial observability constraints, the control does not have access to the full state, but only to some partial observation of it. In this work, we consider three different types of observations which we analyze and compare in their respective sections:

- Section 4.5 considers an observation consisting of only the number of wild males. Formally, the observation space is $\Omega = \mathbb{R}_{\geq 0}$ and the function mapping the state to observations (which in this work is deterministic) is $\mathcal{O}(E(t), M(t), F(t), M_s(t)) = M(t)$.
- Section 4.6 considers an observation consisting of the total number of males. Formally, $\Omega = \mathbb{R}_{\geq 0}$ and $\mathcal{O}(E(t), M(t), F(t), M_s(t)) = M(t) + M_s(t)$.
- Section 4.7 considers an observation consisting of the total number of males and the total number of females. Formally, $\Omega = \mathbb{R}_{\geq 0}^2$ and $\mathcal{O}(E(t), M(t), F(t), M_s(t)) = (M(t) + M_s(t), F(t) + F_s(t))$, where $F_s(t) = F(t)M_s(t)/M(t)$ was defined in Section 4.2.

All of the observations that we consider are quantities that we are able to measure in the real world. This is in contrast to the backstepping control of [2], which requires measurements of E , M and M_s , which is currently not possible in the real world. Besides, these observations may grow very large and thus can span

quite a large range. As such, to enable the neural network to observe large values while still being able to discriminate between smaller values, we input each observation into the control at different orders of magnitude. For instance, $M + M_s$ is inputted at several scales, namely $M + M_s$ becomes $\min(M + M_s, k)$ for k ranging from 5 to $100K$, and similarly for $F + F_s$. We normalize all observations so that they lie within $[0, 1]$. This has proved important to help with convergence during training: normalizing inputs is a common preprocessing technique, and inputting each observation at several scales further helps with training stability without technically adding more inputs to the control. We also consider adding memory of the past observations as an input to the control. The rationale behind is that we hope that with this additional information the neural network will be able to reconstruct in some sense the state and, in other words, internally build a kind of observer that would remain a black box. This approach however, makes it significantly more complex to convert into an explicit feedback.

Action space Our action space $\mathcal{A} = [-1, 1]$ corresponds to a single action $a(t) \in \mathcal{A}$ that is remapped to the range $[0, 10K]$ and then directly inputted into the model equations (4.1)–(4.4) through $u(t) = 5K(a(t) + 1)$. Having the neural network model output a normalized action is a common technique for more robust training, akin to normalization of the inputs (here we output a normalized action, then scale it up).

Initial state distribution We sample the initial state uniformly between 0 and $10K$, which corresponds to physically-realistic values for the components of the state. Namely, $(E(0), M(0), F(0), M_s(0)) \sim \mu = \mathcal{U}([0, 10K]^4)$.

State transition function The state transition function T , that maps a current state and action to a next state, is deterministic in this work and implicitly defined by the ODE system (4.1)–(4.4): we discretize the state $x_t = (E(t), M(t), F(t), M_s(t))$, and at each time step compute the next state through a simple Euler update $x_{t+1} = x_t + \dot{x}_t dt$ (with an abuse of notation, since the update is technically done on each of the four individual components of the state), where the action $u_t = u(t)$ comes in the definition of $\dot{M}_s$. The value for dt is indicated in Section 4.4.4. Thus, the transition function can be written as $T(x_t, u_t) = x_{t+1}$.

Reward function The optimization criterion is usually the most crucial part of the RL process. Our reward function, which we aim to maximize over time as per Eq. (4.17), takes the following form at time step t :

$$r_t = -c_1 (\|E(t)\|_2 + \|M(t)\|_2 + \|F(t)\|_2) - c_2(t) \|M_s(t)\|_2 \quad (4.19)$$

with

$$c_2(t) = \begin{cases} c_3 & \text{if } t < 0.9T, \\ c_3 + c_4 & \text{otherwise,} \end{cases} \quad (4.20)$$

where T is the simulation horizon. In a continuous control framework this would correspond to maximizing the functional

$$J(u) = \int_0^T c_1 \|E(t), M(t), F(t)\|_2 + c_2(t) \|M_s(t)\|_2 dt. \quad (4.21)$$

The specific values we use for c_1 , c_3 and c_4 are indicated in Section 4.4.4, and are derived empirically and heuristically through hyperparameter tuning. The reason for the specific shape of this reward function Eq. (4.19) is as follows: ideally, we would simply penalize all the components of the state equally and have

$$r_t = -C (\|E(t)\|_2 + \|M(t)\|_2 + \|F(t)\|_2 + \|M_s(t)\|_2)$$

for some $C > 0$ in order to drive all the components of the state to zero. This is what we initially tried; however, for practical reasons, we found this objective to be harder to minimize due to the fact that M_s can typically take much larger values than the 3 other component of the state. Indeed, one of the difficulties of this control problem is that M_s should necessarily take high values to be able to bring M , F and E closer to 0, given (4.1)–(4.4), and this results in a delay between the control action and its effect on the mosquito population. A practical interpretation is that the females have no preferences between sterile males and fertile males, hence M_s should be larger than M to have an influence.

As a result, our choice of $c_2(t)$ corresponds to penalizing M_s with less amplitude than we penalize the other 3 components of the state; nevertheless around the end of the simulation we increase the penalty on M_s to encourage convergence to 0. This allows for a high action at the start to grow M_s without excessive penalty, which in turn makes the other components of the state decrease, then a slowly decreasing action so that M_s converges 0. The main challenge is to not decrease M_s too quickly, or the other components of the state would increase again. We empirically found that this reward design led to increased training stability: it is designed in a way to guide the controller's, which is initially random, to a reasonable behavior more quickly. We note that the rewards are also normalized by K to lie within a reasonable range.

Remark 4.4.1. *We note that the reward function during training has access to the full simulation state and may depend on components outside the control input to optimize performance. However, at inference, there is no more reward and the control relies solely on the provided inputs. Actually, one of the key challenges of*

this control problem is precisely that the reward to be maximized during training can depend on the full state, whereas the control itself cannot measure these states. This distinction highlights an important difficulty: the controller must learn to optimize a function it does not fully observe. In fact, the method would be less powerful if we only optimized cost functions that are entirely observable. Thus the RL control has no direct way of fully understanding the reward but still manages to optimize it successfully.

Additionally, in order to artificially reduce the horizon and make training more robust (RL usually suffers from overly long simulations, as it makes optimization a reward over time much more complex), we repeat each action several times, meaning that for each environment step we use the same action to run n_{sims} simulation steps. Finally, the value for γ is indicated in Section 4.4.4.

4.4.3 Method

In our work, we aim to train control policies using RL to regulate a dynamic system described by ordinary differential equations. To do that, we design an environment that simulates the behavior of the ODE system, allowing us to interact with it in a controlled manner. Our approach is summarized in Fig. 4.1. We start by discretizing our system of equations (4.1)-(4.4), and use those dynamics to create a simulation of our model. We use it to create an environment by implementing the observations, actions and rewards described in Section 4.4.2. Using this code, we train an RL agent that learns to maximize the objective function we assign it through many simulations, using the PPO algorithm mentioned in Section 4.4.1. Once the policy has converged, we can evaluate it on any simulation, in particular we can query a control u for any current state $(E, M, F, M_s)^T$. Since we only trained policies with 1 or 2 observations, we can plot the action as a function of the input in 1D or 2D space. This allows us to perform a regression and empirically write a simple explicit control that has the same general shape as the neural network control (see Sections 4.5 and 4.7). Finally, we simulate this explicit control to ensure that it still stabilizes the system, and analyze its properties and its sensitivity with respect to the parameters of the system.

4.4.4 Experiment details

Here we present the experiment details in greater details. To implement the PPO algorithm we use to train our RL policies, we use Stable Baselines 3 [88] (version 1.6.2 in Python 3.8), a popular RL library that provides a collection of state-of-the-art algorithm implementations, as well as various tools for RL research. The models in Sections 4.5, 4.6 and 4.7 are trained for 10 million environment timesteps (or 7 billion simulation timesteps) on 12 CPUs, which takes about 7 hours. During each iteration, we collect 12288 (1024 per CPU) environment steps, then run 5 epochs of optimization with a batch size of 1024. The agent's policy is a fully-connected neural network with 2 hidden layers of 256 neurons each, with tanh non-linearities between each layer, outputting the mean and standard deviation of a normal distribution that is then used to sample the action. More formally, for a given observation vector o_t , the neural network policy outputs $(\mu_t, \sigma_t) = \pi_\theta(o_t)$ and the action is sampled as $a_t \sim \mathcal{N}(\mu_t, \sigma_t)$. We train with a learning rate of 3×10^{-4} , discount coefficient of $\gamma = 0.99$, and all other hyperparameters are left to their default values.

We run each simulation for $T = 1000$ days, with a timestep $dt = 0.01$ days, and each action is repeated $n_{\text{sims}} = 500$ times, meaning that the environment horizon is 200 steps. For each simulation, the initial condition is uniformly sampled between 0 and $10K$: $E(0), M(0), F(0), M_s(0) \sim \mathcal{U}(0, 10K)$. For our reward function, we use coefficients $c_1 = 0.1$, $c_3 = 0.001$ and $c_4 = 0.01$. For the model, we use the parameters given in Table 4.5.

4.5 Feedback RL control using M

In the literature, one of the previous approaches consist in using a linear feedback control that only depends on $M(t)$ [2], that is $u(t) = f(M)$ where f is linear instead of the control law (4.5). They observe that the following linear control feedback law

$$u(t) = \alpha M(t), \quad (4.22)$$

seems to stabilize the system as long as

$$\alpha > \frac{(\beta_E \nu \nu_E - (\nu_E + \delta_E) \delta_F) \delta_s}{(\nu_E + \delta_E) \delta_F}. \quad (4.23)$$

In this section, we use our RL procedure to deduce a potentially nonlinear feedback law that similarly only depends on $M(t)$.

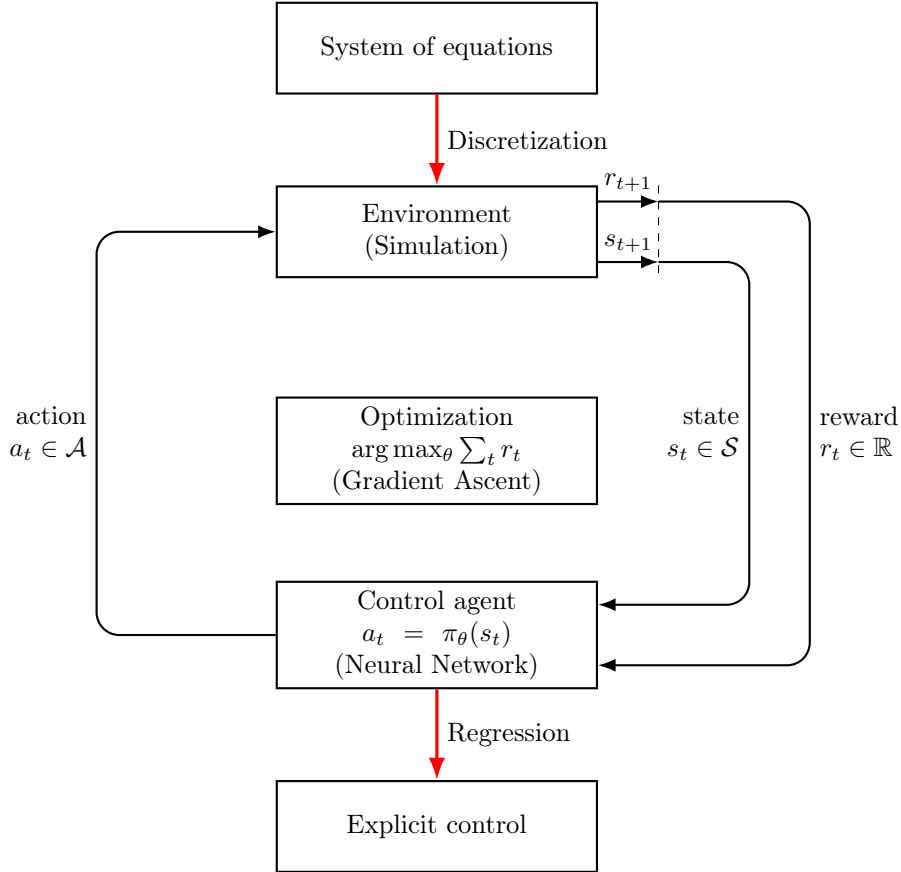


Figure 4.1: Diagram representing the (simplified) procedure by which we simulate our model in an environment that is used to train an RL agent, whose policy we then convert into an explicit control. The policy π_{θ} is modeled by a neural network with parameters θ , which takes a state as an input and outputs an action (or a distribution over actions in the stochastic case). The neural network is then optimized to maximize the sum of rewards it obtains over simulations.

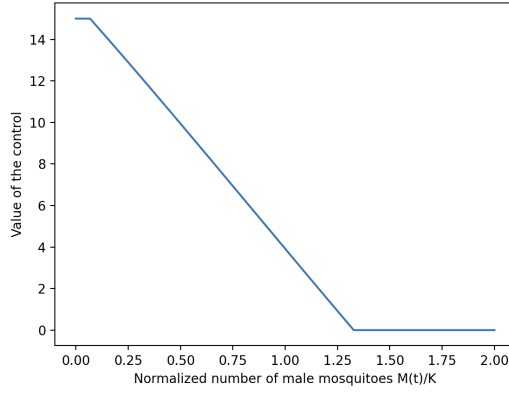


Figure 4.2: Value of the control f_{RL} as a function of $M(t)/K$ when $\alpha_M = 15$ and parameters as in Table 4.5.

We force the feedback control to go to 0 when the number of wild male goes to 0 by searching the control under the form

$$u(t) = \min \left(\min \left(f_{RL} \left(\frac{M(t)}{K} \right), \alpha_M \right) M(t), u_M \right), \quad (4.24)$$

where α_M is a chosen constant, u_M is the maximal value of the control allowed which is dictated by physical constraints, and f_{RL} is the function searched by the RL model. This f_{RL} is searched using the procedure described in Section 4.4 and using the cost function

$$\begin{aligned} J(u) &= - \int_0^T r_t dt \\ r_t &= - \frac{\|E(t), M(t), F(t)\|_2}{K} \\ &\quad + q_1 \left(\frac{M_s(t)}{K} + \max \left(0, \left(\frac{M_s(t)}{K} - 30 \right) \right)^2 \right) \end{aligned} \quad (4.25)$$

where q_1 is a chosen constant, typically much smaller than 1, and T is a chosen horizon. After training, the RL model converges and the optimal numerical f_{RL} obtained has a relatively simple form, shown in Figure 4.2, which happens to be exactly piecewise linear. This allows us to deduce the following nonlinear control feedback law for the system

$$u(t) = \min (\bar{u}(t), u_M) \quad (4.26)$$

with

$$\bar{u}(t) = \min \left[\max \left(\alpha_1 - \alpha_2 \frac{M(t)}{K}, 0 \right), \alpha_M \right] M(t), \quad (4.27)$$

where α_1 and α_2 are positive constants. For the values in Table 4.5 with $\alpha_M = 15$, $\alpha_1 \approx 16$ and $\alpha_2 \approx 12$.

Remark 4.5.1. *The RL control (4.26) tends to be linear when $K \rightarrow +\infty$.*

In Figure 4.3 we present 100 numerical simulations of the closed-loop system when using this explicit control. Each simulation has a different initial condition taken uniformly at random with each component of the state having values in $[0, 10K]$. We can see that all the components of the state of the system converge quickly to 0 after 800 days, while the main components of the system (not sterile males M , females F and eggs E) converge much faster to 0.

In Table 4.1 we show the average and variance of the components of the state E , M , F and M_s at different times over 10000 numerical simulations as well as the maximal absolute value of each component of the state.

4.6 RL control using $M + M_s$

In practice, measuring M is a challenge. In a wild population of mosquitoes, synthetic versions of female insect pheromones are released to attract and capture male insects. This allows for measuring $M + M_s$, however there is no easy way to distinguish between the wild male mosquitoes and the released sterilized male. Another possible measurement can be done by placing simple traps for adult mosquitoes in the wild, then differentiating them based on physical attributes such as their size to separately count total males $M + M_s$ and total females $F + F_s$.

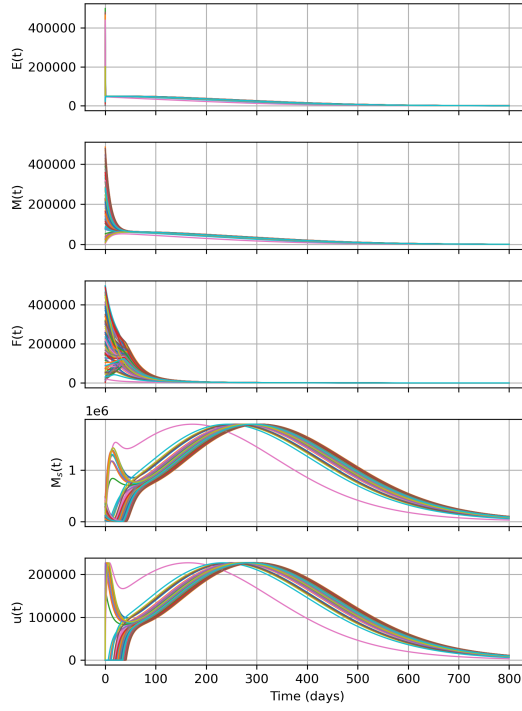


Figure 4.3: Representation of the state in a 100 simulations, with $\alpha_M = 15$, $q_1 = 0.004$, $T = 2000$ days, $u_M = 10K$, and the parameters of Table 4.5. Each simulation corresponds to one color.

	200 days	400 days	800 days	2000 days
average $ E + M + F $	$8.7 \cdot 10^4$	$3.3 \cdot 10^4$	$1.3 \cdot 10^3$	$1.5 \cdot 10^{-2}$
variance $ E + M + F $	$3.6 \cdot 10^7$	$1.4 \cdot 10^7$	$3.7 \cdot 10^4$	$5.5 \cdot 10^{-6}$
average $ M_s $	$1.6 \cdot 10^6$	$1.5 \cdot 10^6$	$8.4 \cdot 10^4$	1.0
variance $ M_s $	$8.2 \cdot 10^9$	$1.3 \cdot 10^{10}$	$1.6 \cdot 10^8$	$2.5 \cdot 10^{-3}$
maximum $ E + M + F $	$9.3 \cdot 10^4$	$3.8 \cdot 10^4$	$1.5 \cdot 10^3$	$1.8 \cdot 10^{-2}$

Table 4.1: Average, variance and maximum of the different components over 100 simulations, with $\alpha_M > \alpha_1 = 13$, $q_1 = 0.004$, $u_M = 10K$ and the parameters of Table 4.5.

This motivates the search, in practice, for a control that would only depend on $M + M_s$ and $F + F_s$. In this section, we consider using only total males $M + M_s$, since this quantity is more easily measured and we compare to the previous section where M is available instead. Interestingly, the situation is much more complicated when $M + M_s$ is available instead of M , as we are presenting in this section.

In [2], it was conjectured that a linearly dependent feedback of $M + M_s$ stabilizes the dynamics at the origin. However this control lacks robustness (see Figure 4.4) with respect to the parameters of the model or of the controller. Indeed the control has the linear form

$$u(t) = \beta(M(t) + M_s(t)), \quad (4.28)$$

and can only work if β satisfy

$$\left(\frac{\beta_{EV}\nu_E - (\nu_E + \delta_E)\delta_F}{\beta_{EV}\nu_E} \right) \delta_s \leq \beta < \delta_s. \quad (4.29)$$

With the parameters set in the Table 4.5, this condition becomes

$$0.118 \leq \beta < 0.12, \quad (4.30)$$

which means that even a tiny imprecision on the model parameters renders the control inefficient.

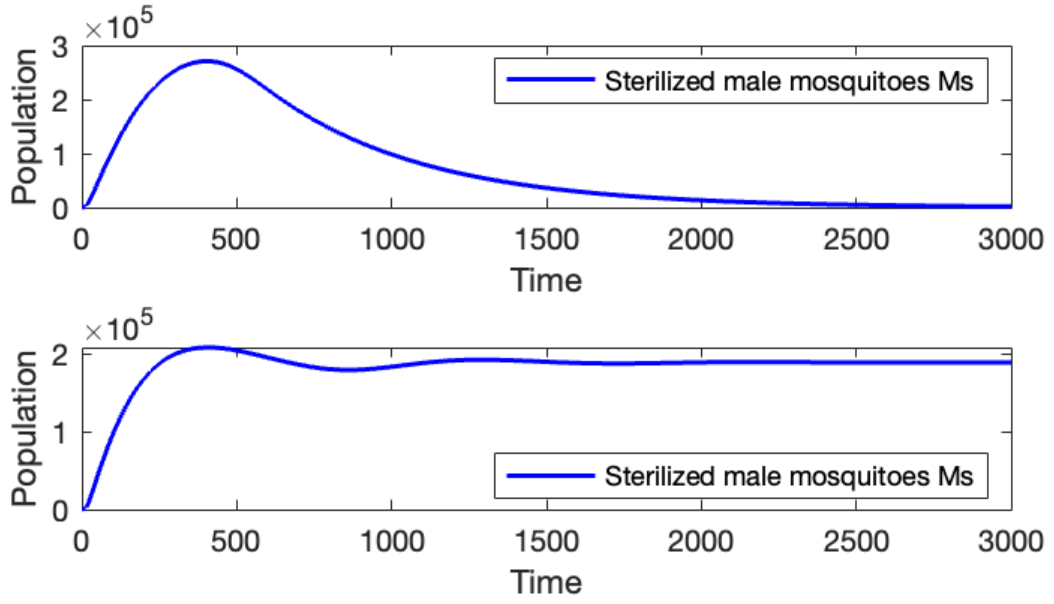


Figure 4.4: Evolution of the sterilized male population as a function of time when applying the feedback law $u(t) = \beta(M(t) + M_s(t))$ with $\beta = 0.118$ (top) or with $\beta = 0.116$ (bottom), over 3000 days.

We tested our RL control procedure in this framework, having only access to $M + M_s$ at the current time t . Even when allowed a nonlinear control, the model does not converge to an efficient control after 1000 iterations of training, with the same setup used to train the other controls. Figure 4.5 illustrates the behavior of the trained control, which outputs an approximately constant control $u(t) \approx 200000$ yielding a stabilization of the populations of mosquitoes, except for the released sterilized male mosquitoes M_s which the control does not manage to reduce with the limited information it has about the state of the system. This demonstrates the lack of robustness mentioned earlier.

Nevertheless, when allowing the control to depend not only on $M + M_s$ at the given time t but also on previous times $s \leq t$, that is to say enabling the control with memory of past values of $M + M_s$, the RL policy converges to what seems to be a robust numerical control. The rationale behind giving the model access to a memory of the measurements is to allow it to derive additional information on the state by internally deriving a kind of observer (that would remain a black box). We train the model by allowing it to measure the state of the system every 7 days and to keep the memory of the 26 most recent measurements (so 6 months of measurements). This measurement frequency corresponds to what is possible in practice [54]. Besides, the control also takes a single action every 7 days. Since this control has strictly more information than the control with only $M + M_s$ at time t , we expect it to perform better as long as the training procedure is stable. It is however not obvious that adding this memory of past observations would be enough to stabilize the system, but Figure 4.6 shows numerically that it appears to be sufficient, across 100 numerical simulations of the closed-loop system using this numerical control with initial conditions chosen at random in $[0, 5K]^4$.

4.7 RL control using $M + M_s$ and $F + F_s$

Using past values of the state can be a challenge to find a mathematical formula from the numerical control. Indeed, when the control depends on the past state, the feedback that is searched is not anymore a function of a finite-dimensional vector but a functional on an infinite dimensional space containing portions of the trajectories (e.g. $(M + M_s(s))_{s \in [t-\tau, t]}$). This makes the symbolic regression a challenge. For this reason we try to find a control using $M + M_s$ and $F + F_s$ only at the current time, that is a control of the form (4.5). These two quantities can be measured in real life, and lead to a much simpler model than the model with memory that has 26 inputs.

A first control In this framework, the RL model is trained as described in Section 4.4 and converges to a numerical control that we represent in Figure 4.7 as a function of $M + M_s$ and $F + F_s$. We see that the plot of the control in linear scale is not really informative (see Fig. 4.7 left). However in log scale the expression of the control seems clearer (see Fig. 4.7 right) and clearly has two parts. In each of them the control seems to be close to a bang-bang control with a thin transition. With a simple regression we approximate this numerical control

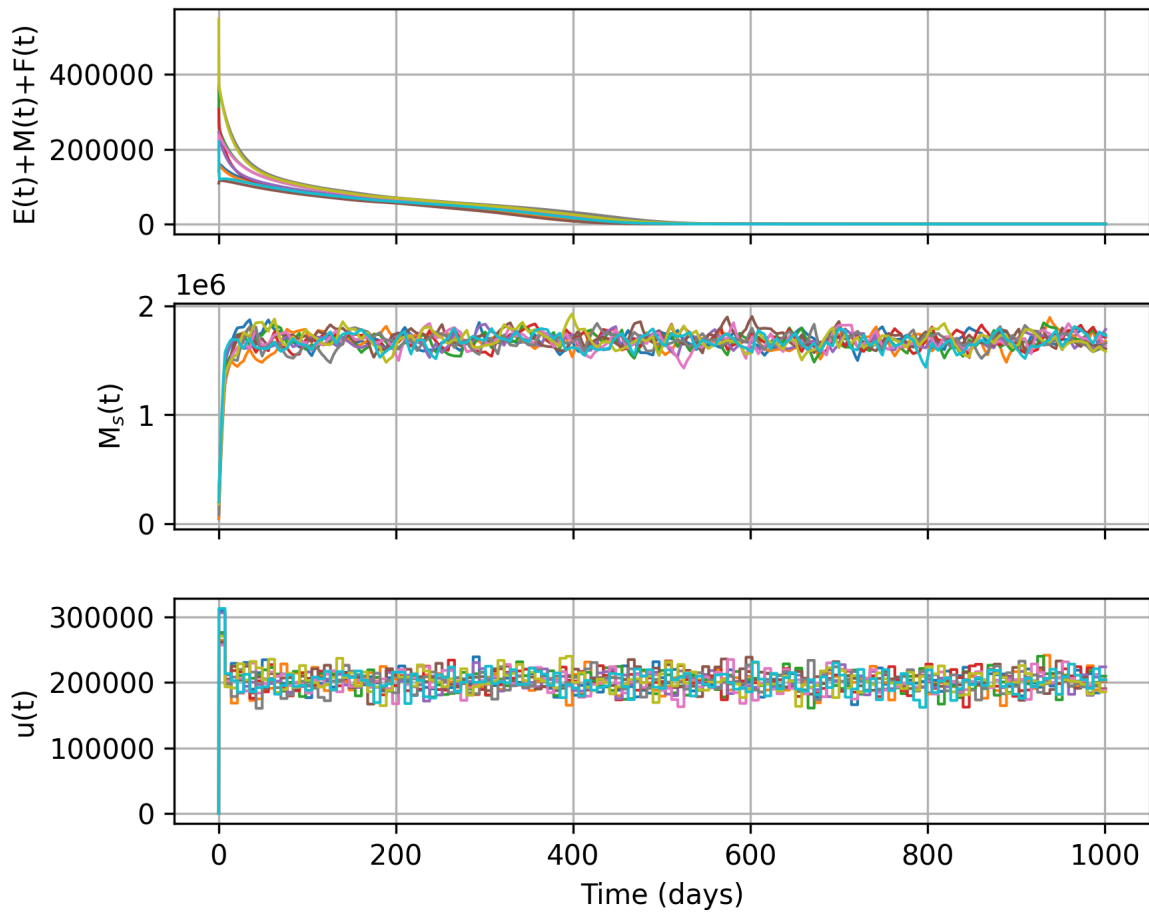


Figure 4.5: Evolution of the population of mosquitoes, represented as $E(t) + M(t) + E(t)$ and $M_s(t)$, and of the control $u(t)$, as a function of time, over 10 simulations with random initial conditions, using an RL control that only has access to $M + M_s(t)$ at the current time t .

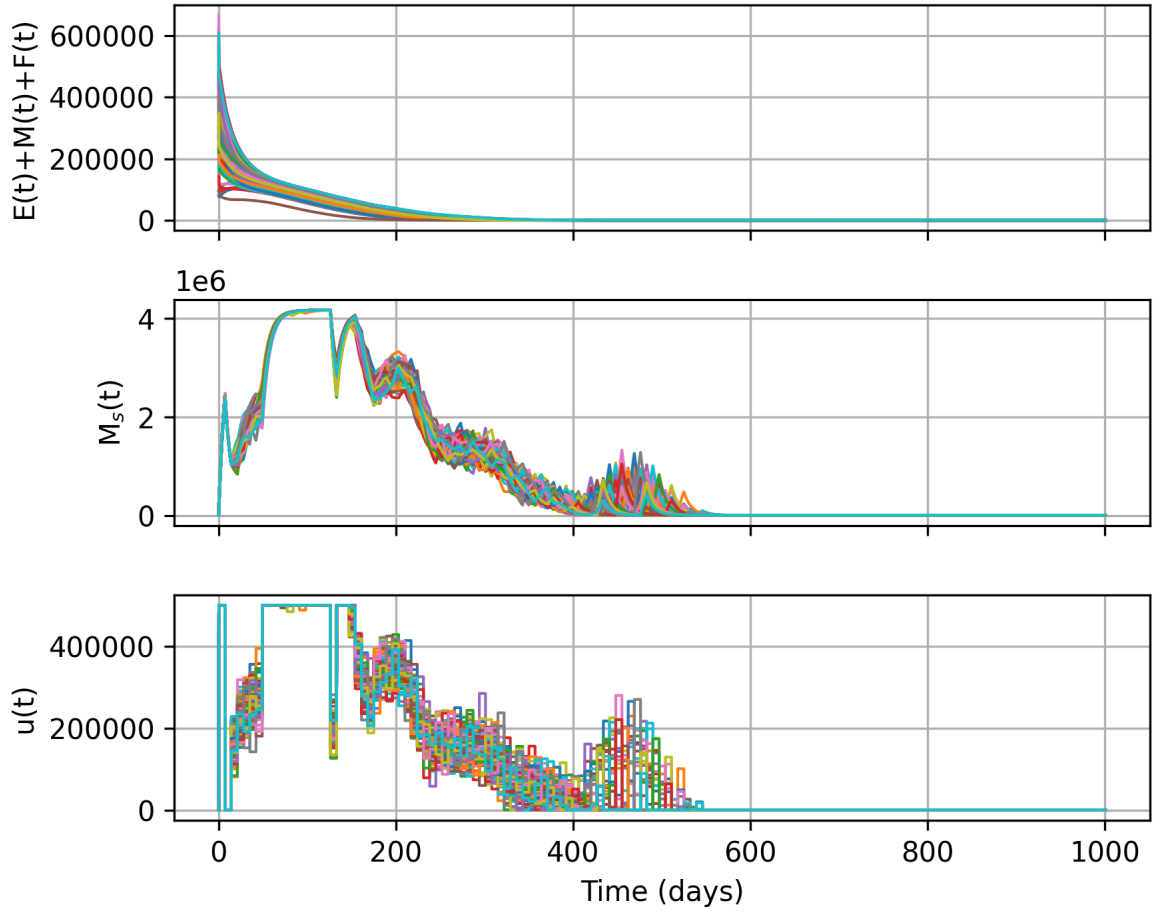


Figure 4.6: Evolution of the population of mosquitoes, represented as $E(t) + M(t) + E(t)$ and $M_s(t)$, and of the control $u(t)$, as a function of time, over 100 simulations with random initial conditions, using an RL control that only has access to measurements of $M + M_s$ over the past 6 months, and every week obtains a new measurement and take a new action.

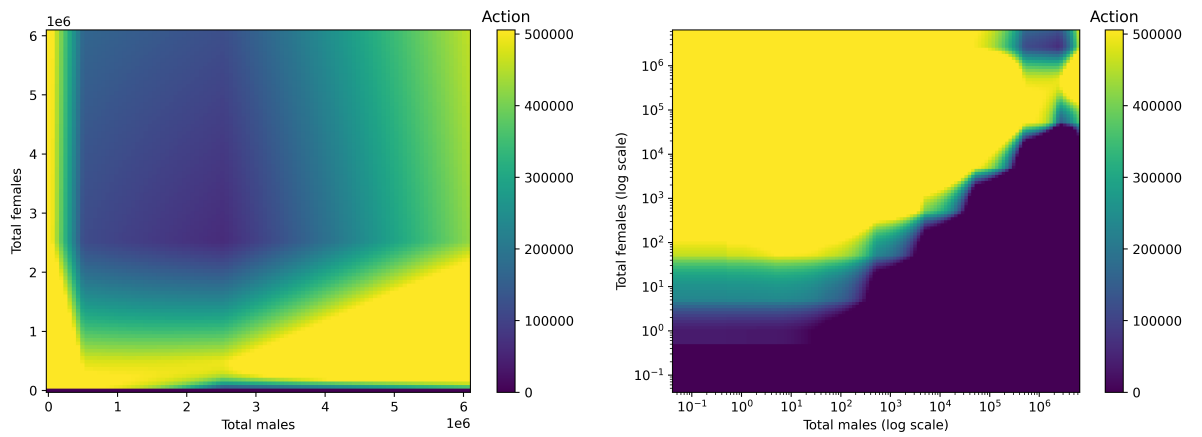


Figure 4.7: Heatmap of the model's action $u(M + M_s, F + F_s)$ as a function of total males and total females, in linear scale (left) and logarithmic scale (right).

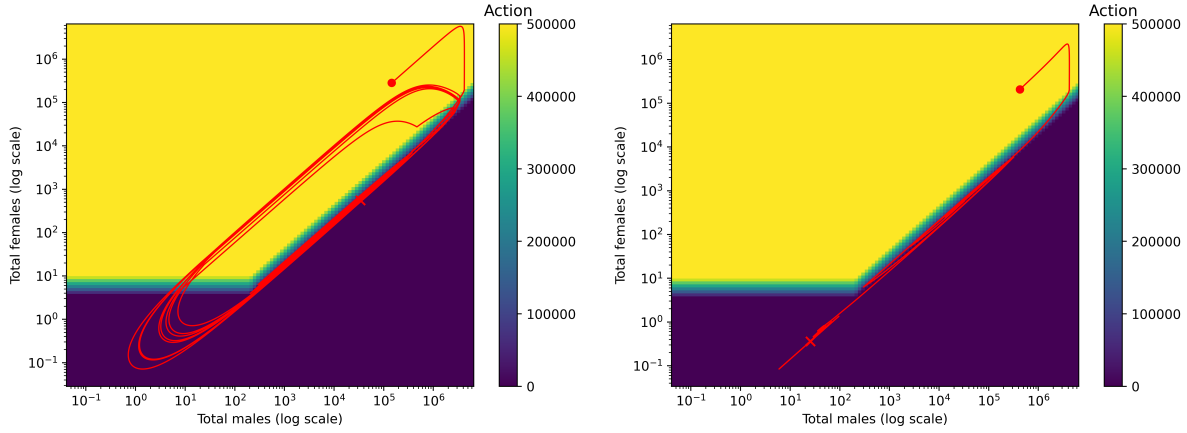


Figure 4.8: Heatmap of the regression model's action $u_{\text{reg}}(M + M_s, F + F_s)$ as a function of total males and total females. A state-space trajectory is plotted in red, with the dot indicating initial state and the cross final state, comparing the no-noise case (left) with the case when a small noise $\mu \sim \mathcal{N}(0, 5)$ is added on top of the action (right).

with

$$u_{\text{reg}}(M + M_s, F + F_s) = \begin{cases} u_{\text{reg}}^{\text{left}} & \text{if } M + M_s < 200, \\ u_{\text{reg}}^{\text{right}} & \text{otherwise,} \end{cases} \quad (4.31)$$

for $M + M_s, F + F_s > 0$, where

$$u_{\text{reg}}^{\text{left}} = \begin{cases} u_{\min} & \text{if } \log \frac{200}{F + F_s} > 4, \\ u_{\max} \left(4 - \log \frac{200}{F + F_s} \right) & \text{if } 4 \geq \log \frac{200}{F + F_s} > 3, \\ u_{\max} & \text{otherwise, and} \end{cases} \quad (4.32)$$

$$u_{\text{reg}}^{\text{right}} = \begin{cases} u_{\min} & \text{if } \log \frac{M + M_s}{F + F_s} > 4, \\ u_{\max} \left(4 - \log \frac{M + M_s}{F + F_s} \right) & \text{if } 4 \geq \log \frac{M + M_s}{F + F_s} > 3, \\ u_{\max} & \text{otherwise.} \end{cases} \quad (4.33)$$

Table 4.2 shows that this explicit control is still able to quickly stabilize the state over a wide range of initial conditions.

During training and testing, the numerical control feedback law includes by default a small (gaussian) noise. This is made to promote some robustness of the control during the training and a good exploration. We tested the mathematical control we derived (given in (4.31)) with and without noise. To our surprise, the control with a small noise does seem to ensure the asymptotic stability, whereas the control without any noise does not seem to. Indeed, without noise, the control seems to have a cyclic behavior and never converges (see Figure 4.8 (left)). When adding a small noise, however, the stability is restored (see Figure 4.8 (right)). The explication to this apparent paradox is that having exactly $u_{\min} = 0$ in one of the branches of the control given in (4.31) is seemingly too strong to allow the model to converge completely. Replacing this value with $u_{\min} > 0$ for a small $u_{\min}$ (typically $u_{\min} = 5$) allows to stabilize the system without noise. In the system with noise, because the control u has to be positive, the noise has the effect of increasing in average the effective value of $u_{\min}$ of the control (4.31), which explains the apparent stabilization. Of course when setting $u_{\min}$ to a small value, the equilibrium that is stabilized is not anymore $(0, 0, 0, 0)$ but $(0, 0, 0, u_{\min}/\delta_s)$ which is very close to it, remains very acceptable in practice compared to the uncontrolled attraction point (especially as only the density of the sterile male mosquitoes does not converge to 0 and, moreover, this density converges to a small value) and answers the mathematical problem described in Section 4.2.

A simpler control We decided to simplify the control found by the RL algorithm. We wanted to see if there is really a need for a different regime when there is only very few mosquitoes. The motivation behind this is that this region does not influence much the cost function that the RL algorithm tries to optimize and the control

might be less precise on this part. This leads to the following simplified formula for the feedback control:

$$v_{\text{reg}}(M + M_s, F + F_s) = \begin{cases} u_{\min} & \text{if } \log \frac{M+M_s}{F+F_s} > \alpha_2, \\ u_{\max} & \text{otherwise,} \end{cases} \quad (4.34)$$

for $M + M_s, F + F_s > 0$, where $\alpha_2 = 4$ is a constant found by regression.

In this case again, the control with $u_{\min} = 0$ and no noise does not seem to ever lead to the convergence of the state $(E(t), M(t), F(t), M_s(t))$, with a notable difference however: there is no purely cyclic behavior and $(E(t), M(t), F(t))$ convergences rapidly to $(0, 0, 0)$. The obstacle to the convergence manifests in large peaks that appear in the control feedback and are increasingly spaced in time, as evidenced in Figure 4.9. When choosing again a small $u_{\min} > 0$ the convergence to the equilibrium is recovered and seems to work for arbitrarily small $u_{\min} > 0$ (see Figure 4.9).

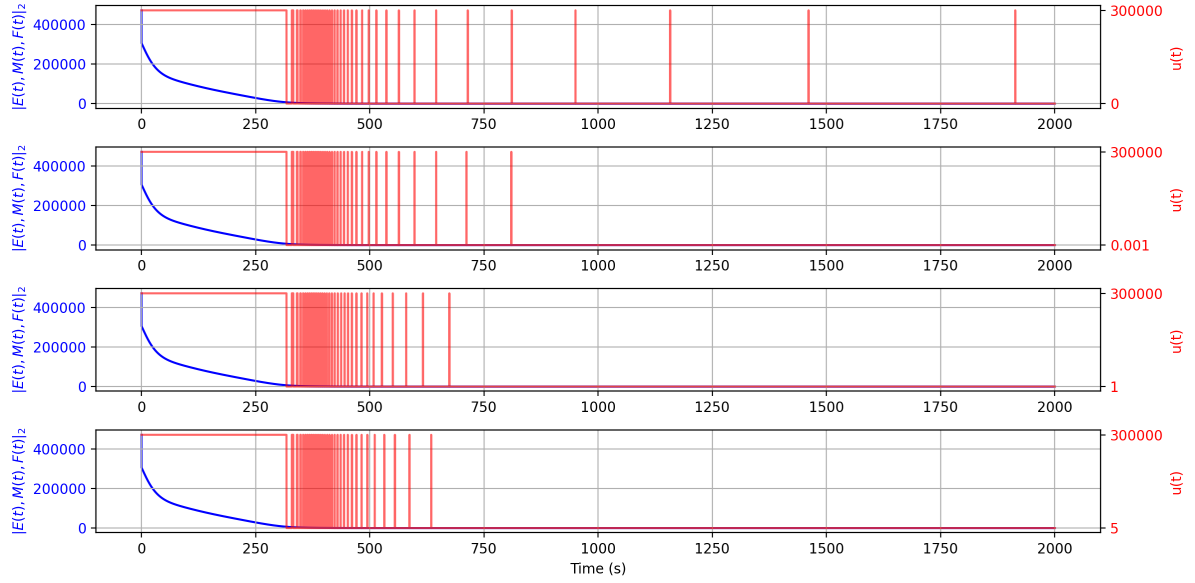


Figure 4.9: Norm of the components of the state $\|E(t), M(t), F(t)\|_2$ (blue) and control v_{reg} (red) as a function of time for different values of $u_{\min}$ (0, 0.001, 1, and 5 respectively from top to bottom) and $u_{\max} = 300000$, over 2000 days and with the same initial condition. When the minimum control $u_{\min}$ is 0, the control occasionally (more and more rarely) outputs the maximum action (corresponding to the spikes) to prevent the norm of the state from going back up. However, when $u_{\min} > 0$, after some time this minimal action is sufficient to stabilize the state around our target (i.e. $(E^*, F^*, M^*) = (0, 0, 0)$) and no more spikes are observed.

We numerically demonstrate that this simplified control seems stabilizing over a wide range of initial conditions, much larger than at training time. Figure 4.10 shows the evolution of the control and components of the state for randomly sampled initial conditions. The control appears effective, quickly stabilizing the state in every simulation. Table 4.3 shows corresponding statistics, illustrating that the state rapidly converges to 0 (with the exception of M_s which takes longer to converge to ensure the other components of the state don't regrow) with very small variance. Besides, we can notice by comparing Table 4.2 and Table 4.3 that the simplified control leads to quicker convergence of the state than the control defined in (4.31)–(4.33).

Sensitivity of the control We previously showed in Figure 4.10 that the control appears effective in a wide range of initial conditions outside of the training distribution. We now investigate the sensitivity of the control with respect to the system parameters (see Table 4.5). Figure 4.11 considers modifying one system parameter while leaving the others to their default values, and shows that the controlled system still converges for a wide range of parameter values, most of which are outside of the training distribution since the training happened with a single set of parameters.

Secondly, we consider randomly perturbing the system parameters simultaneously:

$$\begin{aligned} \hat{\beta}_E &\sim \mathcal{U}(0.65, 9.03), \quad \hat{\nu}_E \sim \mathcal{U}(0.01, 0.35), \quad \hat{\delta}_E \sim \mathcal{U}(0.03, 1.0), \\ \hat{\delta}_F &\sim \mathcal{U}(0.03, 0.1), \quad \hat{\delta}_M \sim \mathcal{U}(0.09, 0.2), \quad \hat{\nu} \sim \mathcal{U}(0.39, 1.0) \end{aligned} \quad (4.35)$$

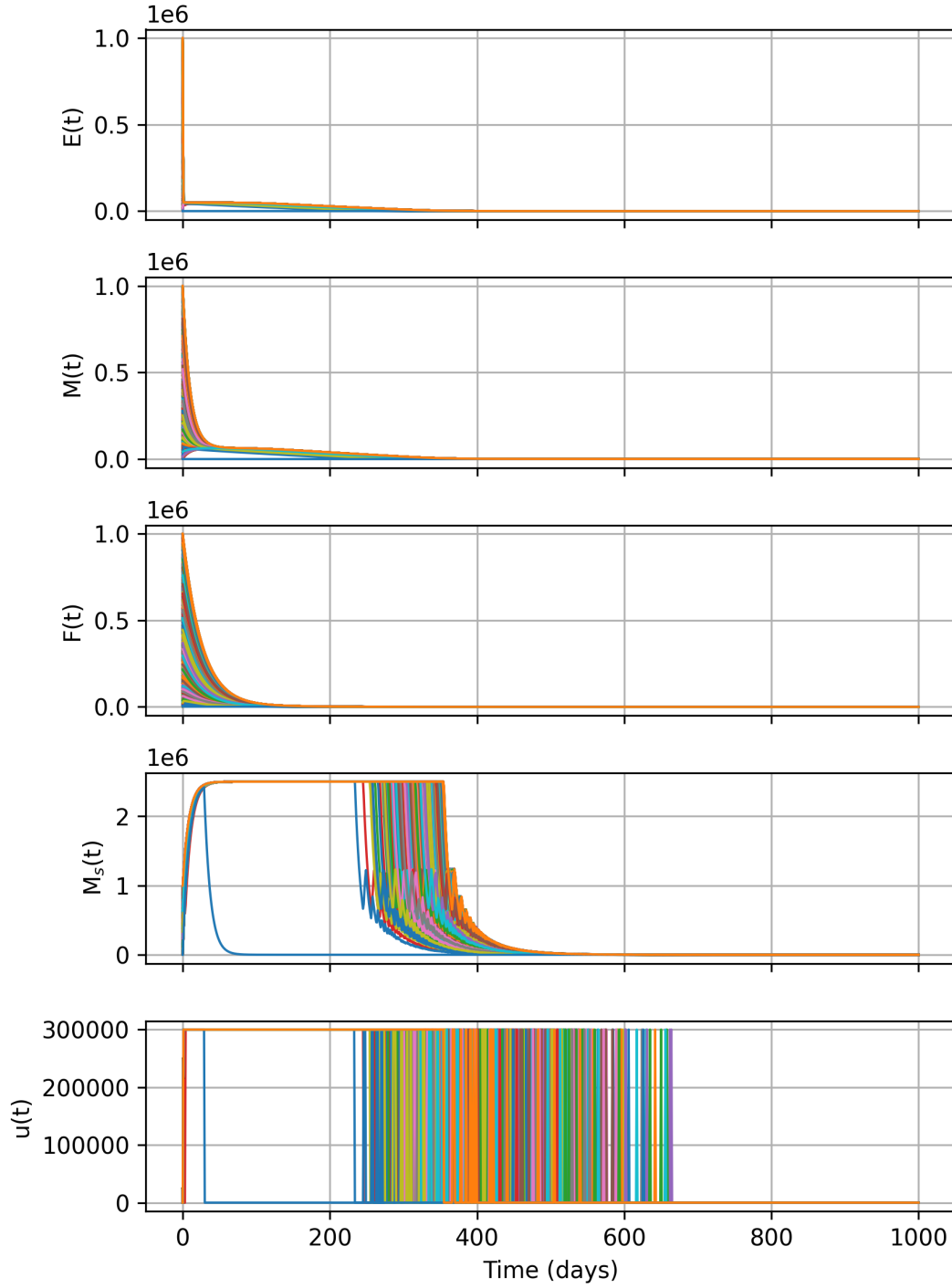


Figure 4.10: Components of the state and control v_{reg} with $u_{\min} = 5$ and $u_{\max} = 300000$ over a duration of 1000 days for 1000 simulations with random initial conditions uniformly sampled in $[0, 20K]^4$ (note that the interval $[0, 5K]^4$ used during training was much smaller). Due to the larger values, we have reduced dt to 10^{-3} for these simulations for numerical stability. Each color correspond to a different simulation.

where $\mathcal{U}(a, b)$ is the uniform distribution on interval $[a, b]$. Table 4.5 shows more details about reasonable and realistic parameter ranges; the parameters tested above have mostly been chosen from the green converging points in Figure 4.11. Table 4.4 shows that over 10,000 random simulations with parameters chosen randomly according

	200 days	400 days	600 days	800 days
average $ E + M + F $	5.08×10^4	8.02×10^3	9.63×10^1	6.00×10^{-1}
variance $ E + M + F $	4.54×10^7	4.01×10^6	1.39×10^3	2.00×10^{-2}
maximum $ E + M + F $	5.90×10^4	1.05×10^4	1.49×10^2	8.00×10^{-1}
average $ M_s $	2.21×10^6	6.94×10^5	1.67×10^4	5.03×10^1
variance $ M_s $	5.31×10^{10}	1.31×10^{10}	4.14×10^7	8.12×10^1
maximum $ M_s $	2.47×10^6	8.22×10^5	2.58×10^4	7.06×10^1

Table 4.2: Statistics over 100 simulations with random initial conditions in $[0, 10K]^4$ using control u_{reg} (see (4.31)) with $u_{\min} = 5$ and $u_{\max} = 300000$ over a duration of 800 days.

	200 days	400 days	600 days	800 days
average $ E + M + F $	4.88×10^4	6.89×10^2	2.47×10^0	2.00×10^{-3}
variance $ E + M + F $	7.58×10^7	7.85×10^4	1.31×10^0	1.61×10^{-6}
maximum $ E + M + F $	5.91×10^4	1.13×10^3	4.37×10^0	6.00×10^{-3}
average $ M_s $	2.50×10^6	1.29×10^5	2.20×10^3	4.17×10^1
variance $ M_s $	3.07×10^{-11}	2.95×10^9	5.20×10^6	2.17×10^{-7}
maximum $ M_s $	2.50×10^6	2.48×10^5	1.08×10^4	4.17×10^1

Table 4.3: Statistics over 100 simulations with random initial conditions in $[0, 10K]^4$ using control v_{reg} (see (4.34)) with $u_{\min} = 5$ and $u_{\max} = 300000$ over a duration of 800 days.

to (4.35), all of them are converging. Overall, this shows that the control trained with given parameters appears to be efficient even for different parameter values within a reasonably large range, which are in line with the uncertainties on the parameters that could be experienced in real life.

Secondly, we consider a more refined model where F_s is the solution of an ODE. Namely, we add the following equation to system (4.1)–(4.4) (which is the last equation of (1) in [102] that we consider with $\beta = +\infty$ and $\mu_F = 0$):

$$\dot{F}_s = \nu\nu_E \frac{M_s}{M + M_s} - \delta_F F_s \quad (4.36)$$

In this case, we investigate whether the previously-obtained control still works. We show numerically that control (4.34) appears to still be stabilizing for the extended system (4.1)–(4.4), (4.36), as evidenced by Figure 4.12.

4.8 Conclusion and perspective

In this paper, we have explored various feedback control strategies for a population control problem involving mosquitoes. We have investigated the use of nonlinear feedback control strategies that depend only on measurements. We have first considered the case where only the wild male mosquito density is measured, then the case where the male mosquito density is measured, and finally the case where both the male and female mosquito densities are measured.

In particular, we have used reinforcement learning to develop a control strategy that depends only on the total number of male and sterile male mosquitoes, as well as the total number of female and sterile female mosquitoes, at a given time. This control strategy appears to be robust to parameter uncertainties and effective, and has the potential to be applied in practice to control mosquito populations and prevent the spread of diseases.

Our results highlight the usefulness of machine learning and control theory in developing effective control strategies for complex dynamical systems, especially when the measurements are limited.

	200 days	400 days	600 days	800 days
average $ E + M + F $	9.71×10^2	6.68×10^0	6.18×10^{-2}	1.02×10^{-4}
variance $ E + M + F $	3.98×10^3	3.38×10^2	5.37×10^0	8.78×10^{-3}
maximum $ E + M + F $	8.15×10^4	3.31×10^4	5.36×10^2	8.75×10^{-1}

Table 4.4: Statistics over 10,000 simulations with random initial conditions in $[0, 10K]^4$ and random system parameters as described in (4.35), using control v_{reg} (see (4.34)) with $u_{\min} = 5$ and $u_{\max} = 300000$ over a duration of 800 days.

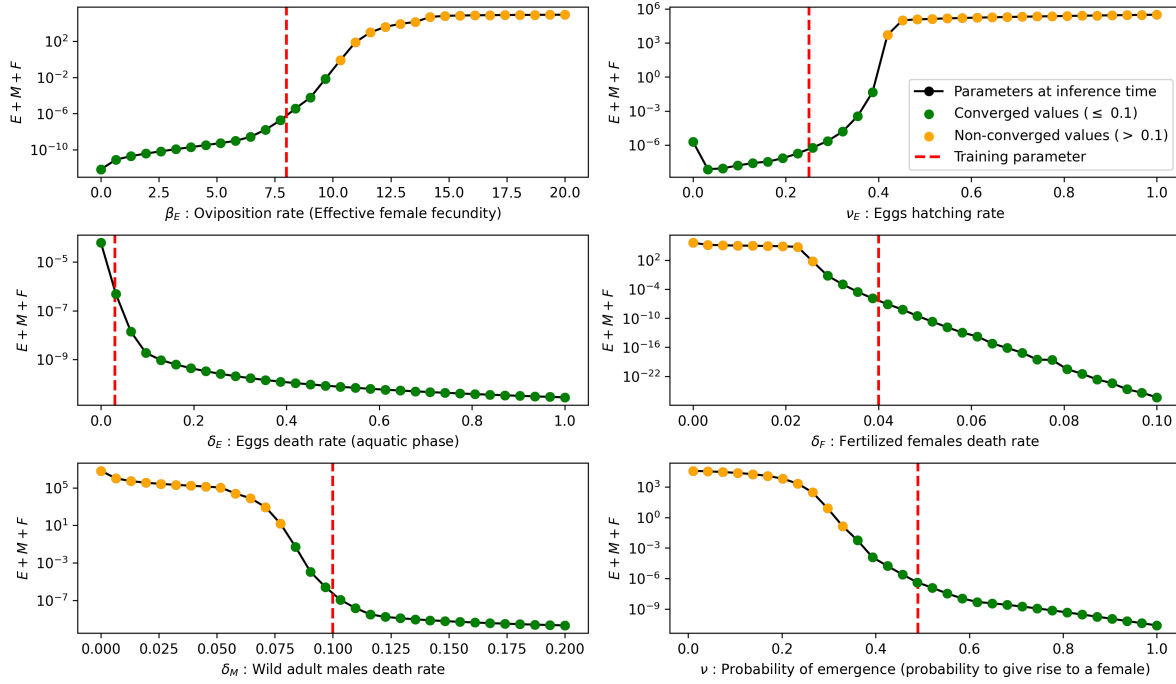


Figure 4.11: Sensitivity of the control to variations in the system parameters. Green points correspond to parameter values for which the system converges at inference time using control v_{reg} (see (4.34)), while orange points correspond to values where the control fails to generalize. The vertical red line shows the unique parameter value at which the controller was trained on; all of the green points correspond to generalization with parameters that the control had never seen during training. When one parameter is changed, all of the others are left to their default values (see Table. 4.5). We note that for the δ_F and δ_M rates, all of the subsequent points up to 1 that are not shown for visibility reasons are also converging.

An interesting question would be to find a Lyapunov function for the closed-loop systems with these feedbacks. This is a challenging question, even when the control feedback is simple [2]. Note that other machine learning based approach are dedicated to this problem of finding a Lyapunov function [6, 5]. Further research in this field could lead to even more powerful techniques for controlling populations of pests and disease vectors.

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During the preparation of this work the author(s) used an LLM in order to correct spellings in the introduction. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

4.9 Typical system parameters

The parameters we use in this work for the (4.1)-(4.4) system can be found in Table 4.4, including their typical value ranges and additional literature regarding how those specific parameters are derived to match realistic values.

¹We consider a larger array of values when assessing the sensitivity of the control with respect to the parameters of the model (see the related paragraph at the end of Section 4.7).

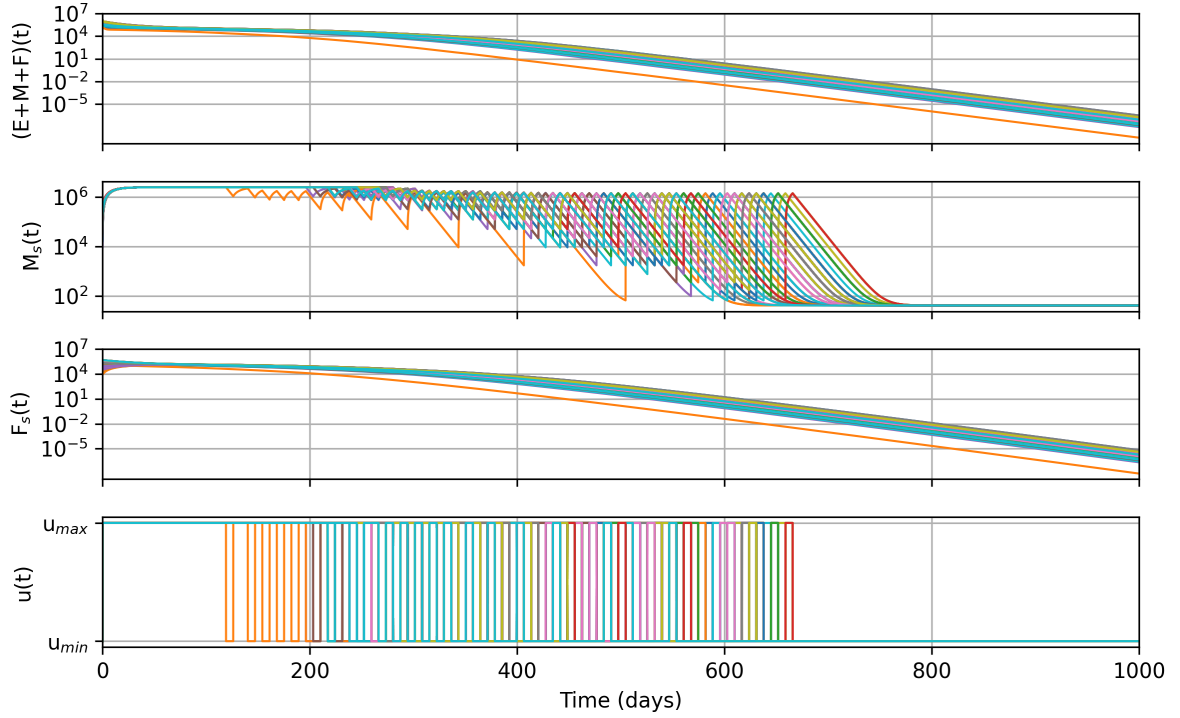


Figure 4.12: Components of the state and control v_{reg} for system (4.1)–(4.4), (4.36) with $u_{\min} = 5$ and $u_{\max} = 300000$ over a duration of 1000 days for 100 simulations with random initial conditions uniformly sampled in $[0, 10K]^4$. Each color correspond to a different simulation. The control v_{reg} takes one new action every 7 days. In all the simulations, E , M , F and F_s appear to converge to 0, M_s converges to a low value (around 40) and the control converges to $u_{\min}$.

	Parameter name	Typical interval	Value in our work ¹	Unit
β_E	Effective fecundity	[7.46, 14.85]	8	Day ⁻¹
ν_E	Hatching parameter	[0.005, 0.25]	0.25	Day ⁻¹
δ_E	Aquatic phase death rate	[0.023, 0.046]	0.03	Day ⁻¹
δ_F	Female death rate	[0.033, 0.046]	0.04	Day ⁻¹
δ_M	Males death rate	[0.077, 0.139]	0.1	Day ⁻¹
δ_s	Sterilized male death rate	-	0.12	Day ⁻¹
ν	Probability of emergence	-	0.49	
K	Environmental capacity for eggs	-	50000	

Table 4.5: Parameters for the system (4.1)–(4.4). This includes typical value ranges, which can be found in a population of *Aedes polynesiensis* in French Polynesia, as well as our chosen values in this work. Further details can be found in [102, 61, 91, 104, 39, 58, 59, 57].

Chapter 5

Some stabilizing linear feedback laws for the sterile insect technique control system

Abstract. The implementation of the Sterile Insect Technique (SIT) to manage a target population has been the focus of numerous recent scientific studies. The present work focuses on a feedback law that depends linearly on the state variables of the SIT control system. We provide both mathematical proof and numerical illustrations demonstrating the global asymptotic stability of the population close to zero when releasing a number of sterile insects proportional to different state variables of the SIT model.

5.1 Introduction

In recent decades, the use of the Sterile Insect Technique (SIT) has been extended to the management of mosquito species such as *Aedes*, *Culex*, and *Anopheles*, which are vectors of various diseases, including *malaria*, *Zika*, *dengue*, and other *arbovirosis*. The mathematical modeling of this method leads to a controlled dynamical system, whose study aims at designing release strategies to achieve specific objectives, such as the asymptotic stabilization of the target population close to extinction, the optimization of release costs, and the robustness of control against environmental variations or system parameter disturbances. The SIT mathematical control system that we consider in this work is introduced in [102]. In some previous works [1, 2, 46], the backstepping method was applied to construct a pivotal feedback law that robustly ensures the stabilisation of the control system to zero. Recently, another pivotal feedback law has been proposed in [29]. The application of these feedback laws requires state measurement, which can be challenging. We address this problem in our previous work by constructing an observer model that requires only the states of sterile males and wild males to estimate all other state variables (see [1]) and by designing a feedback law that depends on the adult population using Reinforcement Learning of the SIT control system [4]. Linear feedback laws are less flexible than non-linear laws, but they are robust and easier to implement in practice. It would be very useful to establish whether the quantity of sterile males released to globally stabilise the population needs to be *proportional* to the density of wild males, total females or eggs, or some other variable state.

Some mathematical progress in this direction was made in our previous work [2], where we aimed at proving that releasing a quantity of sterile males proportional to the wild male population, with a coefficient of proportionality satisfying certain conditions, successfully achieves global stabilisation of the target population to zero. However, proving global stability remains a challenging task.

The structure of this paper is as follows. The mathematical model studied in this paper is presented in Section 5.2, while our main contributions are discussed in Section 5.3. In Section 5.3.1 we present the stability results of the linear feedback law depending on E and $M + M_s$. In Section 5.3.2 we study a linear law including the states E and M . Numerical simulations are provided to illustrate all these results.

5.2 Sterile insecte technique control system

The mathematical modelling of the mosquito life cycle after neglecting the Allee effect is presented in [102]. Its state variables are E which is the density of the aquatic stage (which we will often just designate as eggs but

	Parameter name	Typical interval	Value in our work	Unit
β_E	Effective fecundity	[7.46, 14.85]	8	Day ⁻¹
ν_E	Hatching parameter	[0.005, 0.25]	0.05	Day ⁻¹
δ_E	Aquatic phase death rate	[0.023, 0.046]	0.03	Day ⁻¹
δ_F	Female death rate	[0.033, 0.046]	0.04	Day ⁻¹
δ_M	Male death rate	[0.077, 0.139]	0.1	Day ⁻¹
δ_s	Sterilized male death rate	-	0.12	Day ⁻¹
ν	Probability of emergence	-	0.49	
K	Environmental capacity for eggs	-	50000	

Table 5.1: Parameters for the system (5.4). This includes typical value ranges, which can be found in a population of *Aedes polynesiensis* in French Polynesia, as well as our chosen values in this work. Further details can be found in [102, 61, 91, 104, 39, 58, 59, 57].

which also includes larvae and pupal), M the density of the wild adult males and F the density of adult females mated by wild males. The dynamical system is

$$\begin{aligned}
\dot{E} &= \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \\
\dot{F} &= \nu \nu_E E - \delta_F F, \\
\dot{M} &= (1 - \nu) \nu_E E - \delta_M M,
\end{aligned} \tag{5.1}$$

where $\beta_E > 0$ is the oviposition rate, $\delta_E, \delta_M, \delta_F > 0$ are the death rates for eggs, wild adult males and fertilised females respectively, $\nu_E > 0$ is the hatching rate for eggs, $\nu \in (0, 1)$ is the probability that a pupa will give rise to a female, and $(1 - \nu)$ is therefore the probability that it will give rise to a male. And, to simplify, we assume that females are fertilised as soon as they emerge from the pupal stage. $K > 0$ is the environmental capacity for the aquatic phase. Table 5.1 shows the values of these parameters that we consider in our work. The basic offspring number for this system is given by

$$R = \frac{\beta_E \nu \nu_E}{\delta_F (\delta_E + \nu_E)}. \tag{5.2}$$

This dynamics admits a non-trivial equilibrium called the persistence equilibrium, which we denote by X_{E^*} , where

$$\begin{aligned}
E^* &:= K \left(1 - \frac{1}{R}\right), \\
X_{E^*} &:= (E^*, \frac{\nu \nu_E}{\delta_F} E^*, \frac{(1 - \nu) \nu_E}{\delta_M} E^*)^T.
\end{aligned} \tag{5.3}$$

When sterile males are released into the population according to a control function denoted by u , the compartment of females mated by sterile males, denoted by F_s , is added to the system along with M_s , their evolutionary density. This leads to the following control system introduced in [102] (once the Allee effect is neglected):

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \tag{5.4a}$$

$$\dot{F} = \nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F, \tag{5.4b}$$

$$\dot{M} = (1 - \nu) \nu_E E - \delta_M M, \tag{5.4c}$$

$$\dot{F}_s = \nu \nu_E E \frac{\gamma M_s}{M + \gamma M_s} - \delta_F F_s, \tag{5.4d}$$

$$\dot{M}_s = u(t) - \delta_s M_s. \tag{5.4e}$$

Here $0 < \gamma_s \leq 1$ accounts for the fact that females may have a preference for fertile males and the probability that a female mates with a fertile male is $M/(M + \gamma M_s)$.

Proposition 5.2.1. *Let*

$$t \mapsto (E(t), F(t), M(t), F_s(t), M_s(t))^T$$

be a solution of (5.4), defined at time $t = 0$ and satisfying

$$x_0 = (E(0), F(0), M(0), F_s(0), M_s(0))^T \in \mathbb{R}_+^5.$$

Then it is defined on $[0, +\infty)$. Moreover, if $E(0) \geq K$, then there exists a unique time $t_0 \geq 0$ such that $E(t_0) = K$, and one has:

$$E(t) < K \quad \forall t > t_0. \quad (5.5)$$

Let us consider

$$B_K := \left\{ (E, F, M, F_s, M_s)^T \in [0, +\infty)^5 : E \leq K \right\}. \quad (5.6)$$

Note that for all $u \geq 0$, B_K is positively invariant for (5.4). We define

$$t \mapsto x(x_0, t) := (E(t), F(t), M(t), F_s(t), M_s(t))^T, \quad (5.7)$$

the solution of (5.4) with initial condition x_0 . In this paper, all numerical simulations are performed under the initial condition

$$x_0 = (E(0), F(0), M(0), F_s(0), M_s(0))^T = (X_{E^*}, 0, 0)^T. \quad (5.8)$$

5.3 Linear feedback stabilization

To enhance the robustness of the control strategy, we assume that the sterilization and release process may reduce the longevity of the released sterile males. Consequently we assume

$$\delta_s > \delta_M. \quad (5.9)$$

5.3.1 Linear feedback laws depending on eggs density and total male density

Let us define $\hat{\delta} := \delta_s - \delta_M$. For any $\psi > 0$, we consider a control function:

$$u(x) = \psi(1 - \nu)\nu_E E + \hat{\delta}(M + M_s). \quad (5.10)$$

After computing the non-trivial equilibrium point, we obtain the following condition on ψ :

$$\psi \geq \frac{R-1}{\gamma} - \frac{\hat{\delta}}{\delta_M}, \quad (5.11)$$

ensuring that the extinction equilibrium $\mathbf{0}$ is the only equilibrium of system (5.4) with the feedback law (5.10). In the following proposition we present a result concerning the closed-loop system, which is important in proving the asymptotic stability result above.

Proposition 5.3.1. *Let $\psi > 0$. Let*

$$t \in [0, +\infty) \mapsto x(x_0, t) = (E(t), F(t), M(t), F_s(t), M_s(t))^T$$

be a solution of system (5.4) with the feedback law (5.10) and initial condition $x_0 \in B_K$. Then, for all $t \geq \frac{\psi}{\delta}$,

$$M_s(t) \geq \psi M(t).$$

Proof. Let us consider $t \mapsto (E(t), F(t), M(t), F_s(t), M_s(t))^T$ solution of the system (5.4) with the feedback law (5.10) for initial data in B_K . Substituting (5.10) in (5.4e), gives

$$\dot{M}_s = \psi(1 - \nu)\nu_E E + \hat{\delta}M - \delta_M M_s.$$

For all $t \geq 0$ recall that

$$M(t) = M^0 e^{-\delta_M t} + (1 - \nu)\nu_E e^{-\delta_M t} \int_0^t E(s) e^{\delta_M s} ds. \quad (5.12)$$

$$\begin{aligned}
M_s(t) &= M_s^0 e^{-\delta_s t} + \hat{\delta} M^0 t e^{-\delta_M t} \\
&\quad + \hat{\delta}(1-\nu)\nu_E e^{-\delta_M t} \int_0^t (t-s)E(s)e^{\delta_M s} ds \\
&\quad + \psi(1-\nu)\nu_E e^{-\delta_M t} \int_0^t E(s)e^{\delta_M s} ds.
\end{aligned} \tag{5.13}$$

We deduce that $M_s(t) - \psi M(t) \geq M^0 e^{-\delta_M t} (\hat{\delta} t - \psi)$. Hence for all $t \geq \frac{\psi}{\hat{\delta}}$, $M_s(t) \geq \psi M(t)$ and this concludes the proof. $\square$

Remark 5.3.1. *This result shows that when the sterile male is released according to the feedback law (5.10) in the system (5.4), for all $t \geq \frac{\psi}{\hat{\delta}}$, the density of the sterile male becomes greater than the male density for some chosen proportionality coefficient ψ . Referring to [29, 2], if ψ is greater than some value depending on the number of offspring R , the population converges to the extinction equilibrium.*

Let $\mathcal{H} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the function $\mathcal{H} : p \mapsto \frac{R}{1+\gamma p}$. Note that $\psi > \frac{R-1}{\gamma}$ implies $\mathcal{H}(\psi) < 1$.

Theorem 5.3.1. *We assume that*

$$\psi > \frac{R-1}{\gamma}. \tag{5.14}$$

Then, $\mathbf{0}$ is globally exponentially stable for the system (5.4) with the feedback law (5.10) in $[0, +\infty)^5$. More precisely, for every $c_r \in [0, c_e)$ with

$$c_e := \min \left\{ \frac{\nu\nu_E(1-\mathcal{H}(\psi))}{1+\mathcal{H}(\psi)}, \delta_M, \frac{\beta_E\delta_F(1-\mathcal{H}(\psi))}{2}, \delta_F, \delta_s, (\nu_E + \delta_E), \delta_F \right\}, \tag{5.15}$$

there exists $C > 0$ such that, for every solution

$$t \in [0, +\infty) \mapsto x(x_0, t) = (E(t), F(t), M(t), F_s(t), M_s(t))^T$$

of (5.4) with the feedback law (5.10) and $x_0 \in [0, +\infty)^5$, one has

$$\|x(x_0, t)\| \leq C\|x_0\|e^{-c_r t} \quad \forall t \geq 0. \tag{5.16}$$

Proof. We consider

$$t \mapsto x(x_0, t) := (E(t), F(t), M(t), F_s(t), M_s(t))^T$$

the solution of (5.4) for some $x_0 \in B_K$. Let us consider the subsystem for (5.4a), (5.4b), and (5.4c). Note that this subsystem does not depend on (5.4d) and (5.4e). We denote by $t \mapsto z(t) := (E(t), F(t), M(t))^T$ its solution. We will prove that when ψ satisfies condition (5.14), any solution $t \mapsto z(t)$ converges exponentially to $(0, 0, 0)^T$ with some convergence rate. To prove this result, we consider the function V defined by (see [2]):

$$\begin{aligned}
V : z \in [0, +\infty)^3 &\mapsto V(z) \in \mathbb{R}_+, \\
V(z) &:= \frac{1+\mathcal{H}(\psi)}{1-\mathcal{H}(\psi)}E + M + \frac{2\beta_E}{\delta_F(1-\mathcal{H}(\psi))}F.
\end{aligned} \tag{5.17}$$

We observe that this function satisfies the following condition, which is important for proving the exponential stability:

$$Q_1\|z\| \leq V(z) \leq Q_2\|z\| \quad \forall z \geq (0, +\infty)^3. \tag{5.18}$$

where using (5.14)

$$Q_1 := \min \left\{ \frac{1+\mathcal{H}(\psi)}{1-\mathcal{H}(\psi)}, 1, \frac{2\beta_E}{\delta_F(1-\mathcal{H}(\psi))} \right\} > 0, \tag{5.19}$$

$$Q_2 := \max \left\{ \frac{1+\mathcal{H}(\psi)}{1-\mathcal{H}(\psi)}, 1, \frac{2\beta_E}{\delta_F(1-\mathcal{H}(\psi))} \right\} > 0. \tag{5.20}$$

Thus,

$$V \text{ is of class } \mathcal{C}^1, \tag{5.21}$$

$$V(z) > V((0, 0, 0)^T) = 0, \quad \forall z \in [0, +\infty)^3 \setminus \{(0, 0, 0)^T\}, \tag{5.22}$$

$$V(z) \rightarrow +\infty \text{ as } \|z\| \rightarrow +\infty. \tag{5.23}$$

Thus, V is a candidate to be a Lyapunov function. It remains to prove that there exists a constant $c > 0$ such

that $\dot{V}(z(t)) \leq -cV(z(t))$ for all $t \geq 0$. We start by computing its derivative:

$$\begin{aligned}\dot{V}(z) &= \nabla V(z) \cdot g(z) \\ &= \begin{pmatrix} \frac{1+\mathcal{H}(\psi)}{1-\mathcal{H}(\psi)} \\ 1 \\ \frac{2\beta_E}{\delta_F(1-\mathcal{H}(\psi))} \end{pmatrix} \cdot \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) - (\delta_E + \nu_E)E \\ (1-\nu)\nu_E E - \delta_M M \\ \frac{\nu\nu_E M}{M+\gamma M_s} E - \delta_F F \end{pmatrix}. \\ \dot{V}(z) &= \beta_E \frac{1+\mathcal{H}(\psi)}{1-\mathcal{H}(\psi)} F - \delta_M M \\ &\quad - \frac{1+\mathcal{H}(\psi)}{1-\mathcal{H}(\psi)} \frac{\beta_E}{K} EF - \frac{2\beta_E \delta_F}{\delta_F(1-\mathcal{H}(\psi))} F \\ &\quad - (\nu_E + \delta_E) \frac{1+\mathcal{H}(\psi)}{1-\mathcal{H}(\psi)} E + (1-\nu)\nu_E E \\ &\quad + \frac{2\beta_E \nu \nu_E}{\delta_F(1-\mathcal{H}(\psi))} \frac{M}{M+\gamma M_s} E.\end{aligned}$$

From Proposition 5.14, we deduce that for all $t \geq T_0 := \frac{\psi}{\delta}$, we have

$$\frac{M}{M+\gamma M_s} \leq \frac{1}{1+\gamma\psi}. \quad (5.24)$$

Thus, for all $t \geq T_0$,

$$\dot{V}(z) \leq -\beta_E F - \delta_M M - \frac{1+\mathcal{H}(\psi)}{1-\mathcal{H}(\psi)} \frac{\beta_E}{K} EF - (\nu\nu_E + \delta_E)E.$$

Using (5.14), we deduce the existence of $c_a > 0$ such that for all $t \geq T_0$

$$\dot{V}(z) \leq -c_a V(z), \quad \forall z \in [0, +\infty)^3, \quad (5.25)$$

where

$$c_a = \min \left\{ \frac{\nu\nu_E(1-\mathcal{H}(\psi))}{1+\mathcal{H}(\psi)}, \delta_M, \frac{\beta_E \delta_F(1-\mathcal{H}(\psi))}{2} \right\}.$$

We deduce from (5.18) that there exists $C > 0$ such that

$$\|z(t)\| \leq C\|z(T_0)\|e^{-c_a(t-T_0)}, \quad \forall t \geq T_0. \quad (5.26)$$

Note that for all $t \in [0, T_0]$, from the system (5.4) with the feedback law (5.10), we deduce that

$$\frac{d}{dt}\|z(t)\| \leq C\|z(t)\|. \quad (5.27)$$

In (5.27) and until the end of this proof, C denotes constants which may vary from place to place but are independent of x . Hence

$$\|z(T_0)\| \leq C\|z(0)\|. \quad (5.28)$$

Thus, there exists $C > 0$ such that

$$\|z(t)\| \leq C\|z(0)\|e^{-c_a t}, \quad \forall t \geq 0.$$

From (5.4e) and (5.10), we deduce that if $c_a = \delta_M$, then

$$\begin{aligned}M_s(t) &\leq M_s^0 e^{-\delta_s t} + \hat{\delta} M^0 t e^{-\delta_M t} \\ &\quad + \hat{\delta} C\|z(0)\|(1-\nu)\nu_E t^2 e^{-\delta_M t} \\ &\quad + \psi C\|z(0)\|(1-\nu)\nu_E t e^{-\delta_M t}.\end{aligned}$$

Otherwise,

$$\begin{aligned}
M_s(t) &\leq M_s^0 e^{-\delta_s t} + \hat{\delta} M^0 t e^{-\delta_M t} \\
&\quad + \frac{\hat{\delta} C \|z(0)\| (1-\nu) \nu_E}{(\delta_M - c_a)^2} \\
&\quad (e^{-c_a t} - e^{-\delta_M t} - (\delta_M - c_a) t e^{-\delta_M t}) \\
&\quad + \frac{\psi C \|z(0)\| (1-\nu) \nu_E}{\delta_M - c_a} (e^{-c_a t} - e^{-\delta_M t}).
\end{aligned} \tag{5.29}$$

We deduce that for $c_m < c_a$ there exists $C > 0$ independent of x such that for all $t \geq 0$

$$M_s(t) \leq C \|x(0)\| e^{-c_m t}. \tag{5.30}$$

From (5.4d), we deduce that

$$F_s(t) \leq F_s^0 e^{-\delta_F t} + \frac{C \nu \nu_E \|z(0)\|}{\delta_F - c_a} (e^{-c_a t} - e^{-\delta_F t}). \tag{5.31}$$

We deduce that for $c_m < \min\{c_a, \delta_F\}$, there exists $C > 0$ independent of x such that for all $t \geq 0$

$$F_s(t) \leq C \|x(0)\| e^{-c_m t}. \tag{5.32}$$

Hence, for all $x_0 \in B_K$, there exists a constant $C > 0$ such that all solution $t \mapsto x(x_0, t)$ of (5.4) with the feedback law (5.10) satisfies

$$\|x(x_0, t)\| \leq C \|x_0\| e^{-c_m t}, \quad \forall t \geq 0. \tag{5.33}$$

If $x_0 \notin B_K$ then there exists a unique time $t_0 \geq 0$ such that $E(t_0) = K$, and one has:

$$E(t) < K \quad \forall t > t_0. \tag{5.34}$$

For all $t \in [0, t_0]$ one has

$$\begin{aligned}
E(t) &\leq E^0 e^{-(\delta_E + \nu_E)t}, \\
F(t) &\leq F^0 e^{-\delta_F t} + \frac{\nu \nu_E E^0}{\delta_F - (\delta_E + \nu_E)} (e^{-(\nu_E + \delta_E)t} - e^{-\delta_F t}), \\
M(t) &\leq M^0 e^{-\delta_M t} + \frac{(1-\nu) \nu_E E^0}{\delta_M - \delta_F} (e^{-\delta_F t} - e^{-\delta_M t}), \\
F_s(t) &\leq F_s^0 e^{-\delta_F t} + \frac{\nu \nu_E E^0}{\delta_F - (\delta_E + \nu_E)} (e^{-(\nu_E + \delta_E)t} - e^{-\delta_F t}),
\end{aligned} \tag{5.35}$$

Substitution (5.35) and (5.10) in (5.4e), we prove that for every

$$c_w < \min\{\delta_M, \delta_F, \delta_s, (\nu_E + \delta_E)\}$$

there exists $C > 0$ such that

$$\|M_s(t)\| \leq C \|x_0\| e^{-c_w t}, \quad \forall t \in [0, t_0], \tag{5.36}$$

and therefore, using also (5.35)

$$\|x(x_0, t)\| \leq C \|x_0\| e^{-c_w t}, \quad \forall t \in [0, t_0]. \tag{5.37}$$

Hence we conclude that when ψ satisfies (5.14), for all $c_r < c_e$ with c_e defined in (5.15), there exists $C > 0$ such that for all $x_0 \in [0, +\infty)^5$

$$\|x(x_0, t)\| \leq C \|x_0\| e^{-c_r t} \quad \forall t \geq 0. \tag{5.38}$$

□

The biological interpretation of this result is as follows: assuming that females exhibit no preference for fertile males (i.e. $\gamma = 1$), Theorem 5.3.1 guarantees that the following linear control law

$$u(x) = 2R(1-\nu)\nu_E E + (\delta_s - \delta_M)(M + M_s) \tag{5.39}$$

globally asymptotically stabilizes the sterile control system to $\mathbf{0}$. We run the dynamics as long as $E > 1$ and the result is shown in Figure 5.1.

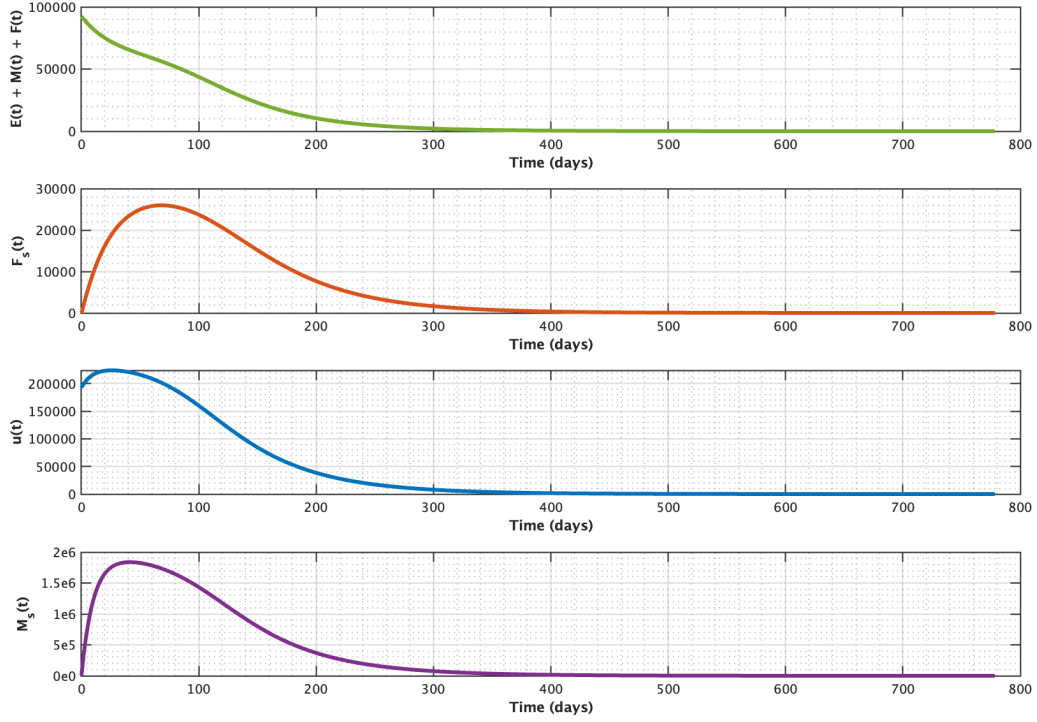


Figure 5.1: Plot of the system (5.4) with the feedback law (5.10) for $\psi = 2R$. According to the parameters fixed in Table 5.1, we have $R = 76.56$

5.3.2 Linear feedback laws depending on aquatic phase and wild male densities

We consider the control function

$$u(x) = \alpha M + (1 - \nu)\nu_E \sigma E. \quad (5.40)$$

Proposition 5.3.2. *Let*

$$t \mapsto x(x_0, t) = (E(t), F(t), M(t), F_s(t), M_s(t))^T$$

be a solution to the system (5.4) with the feedback law (5.40) and initial data $x_0 \in B_K$. For any $\sigma > 0$, for all $\alpha > \sigma\hat{\delta}$, one has $\forall t \geq T_e = \frac{-\ln(1 - \frac{(\delta_s - \delta_M)\sigma}{\alpha})}{\alpha}$

$$M_s(t) \geq \sigma M(t). \quad (5.41)$$

Proof. Let

$$t \mapsto x(x_0, t) = (E(t), F(t), M(t), F_s(t), M_s(t))^T$$

be a solution of system (5.4) with the feedback law (5.40) and initial data $x_0 \in B_K$. for all $t \geq 0$

$$\begin{aligned} M_s(t) &= M_s^0 e^{-\delta_s t} + \frac{\alpha M^0}{\hat{\delta}} e^{-\delta_M t} (1 - e^{-\hat{\delta} t}) + (1 - \nu)\nu_E \sigma e^{-\delta_s t} \int_0^t E(s) e^{\delta_s s} ds \\ &\quad + \frac{(1 - \nu)\nu_E \alpha}{\hat{\delta}} e^{-\delta_M t} \int_0^t E(s) e^{\delta_M s} ds - \frac{(1 - \nu)\nu_E \alpha}{\hat{\delta}} e^{-\delta_s t} \int_0^t E(s) e^{\delta_s s} ds, \end{aligned}$$

and

$$\sigma M(t) = \sigma M^0 e^{-\delta_M t} + \sigma(1 - \nu)\nu_E e^{-\delta_M t} \int_0^t E(s) e^{\delta_M s} ds.$$

Hence,

$$\begin{aligned} M_s(t) - \sigma M(t) &\geq M^0 e^{-\delta_M t} \left(\frac{\alpha}{\hat{\delta}} (1 - e^{-\hat{\delta} t}) - \sigma \right) \\ &\quad + (1 - \nu) \nu_E \int_0^t E(s) \left(e^{-(t-s)\delta_s} - e^{-(t-s)\delta_M} \right) \left(\sigma - \frac{\alpha}{\hat{\delta}} \right) ds. \end{aligned}$$

From (5.9), $\hat{\delta} > 0$ and

$$\int_0^t E(s) \left(e^{-(t-s)\delta_s} - e^{-(t-s)\delta_M} \right) \left(\sigma - \frac{\alpha}{\hat{\delta}} \right) ds \geq 0,$$

is equivalent to $\alpha \geq \sigma \hat{\delta}$. Note that if $\alpha = \sigma \hat{\delta}$, $\frac{\alpha}{\hat{\delta}} (1 - e^{-\hat{\delta} t}) - \sigma < 0$. For $\alpha > \sigma \hat{\delta}$, $\frac{\alpha}{\hat{\delta}} (1 - e^{-\hat{\delta} t}) - \sigma \geq 0$ is equivalent to $t > \frac{-\ln(1 - \frac{(\delta_s - \delta_M)\sigma}{\alpha})}{\alpha}$. We conclude that by choosing $\alpha > \sigma \hat{\delta}$, for any $\sigma > 0$, one has $M_s(t) \geq \sigma M(t)$ for all $t \geq T_e = \frac{-\ln(1 - \frac{(\delta_s - \delta_M)\sigma}{\alpha})}{\alpha}$.

This concludes the proof. $\square$

Theorem 5.3.2. *We assume*

$$\sigma > \frac{R-1}{\gamma}. \quad (5.42)$$

Then, $\mathbf{0}$ is globally exponentially stable for the system (5.4) with the feedback law (5.40) in $[0, +\infty)^5$. More precisely, for every $c_p \in [0, c_b)$ with

$$c_b := \min \left\{ \frac{\nu \nu_E (1 - \mathcal{H}(\sigma))}{1 + \mathcal{H}(\sigma)}, \delta_M, \frac{\beta_E \delta_F (1 - \mathcal{H}(\sigma))}{2}, \delta_F, \delta_s, (\nu_E + \delta_E), \delta_F \right\}, \quad (5.43)$$

there exists $C > 0$ such that, for every solution

$$t \in [0, +\infty) \mapsto x(x_0, t) = (E(t), F(t), M(t), F_s(t), M_s(t))^T$$

of (5.4) with the feedback law (5.10) and $x_0 \in [0, +\infty)^5$, one has

$$\|x(x_0, t)\| \leq C \|x_0\| e^{-c_b t} \quad \forall t \geq 0. \quad (5.44)$$

Proof. The proof of this theorem is similar to the proof of Theorem 5.3.1 using Proposition 5.3.2 instead of Proposition 5.3.1. $\square$

The biological interpretation of this result is as follows: assuming that females exhibit no preference for fertile males (i.e. $\gamma = 1$), Theorem 5.3.2 guarantees that the linear control law

$$u(x) = 4R(\delta_s - \delta_M)M + 2R(1 - \nu)\nu_E E, \quad (5.45)$$

globally asymptotically stabilizes the target population to $\mathbf{0}$ when the death rate of sterile male released is higher than that of wild males. We run the system as long as $E > 1$.

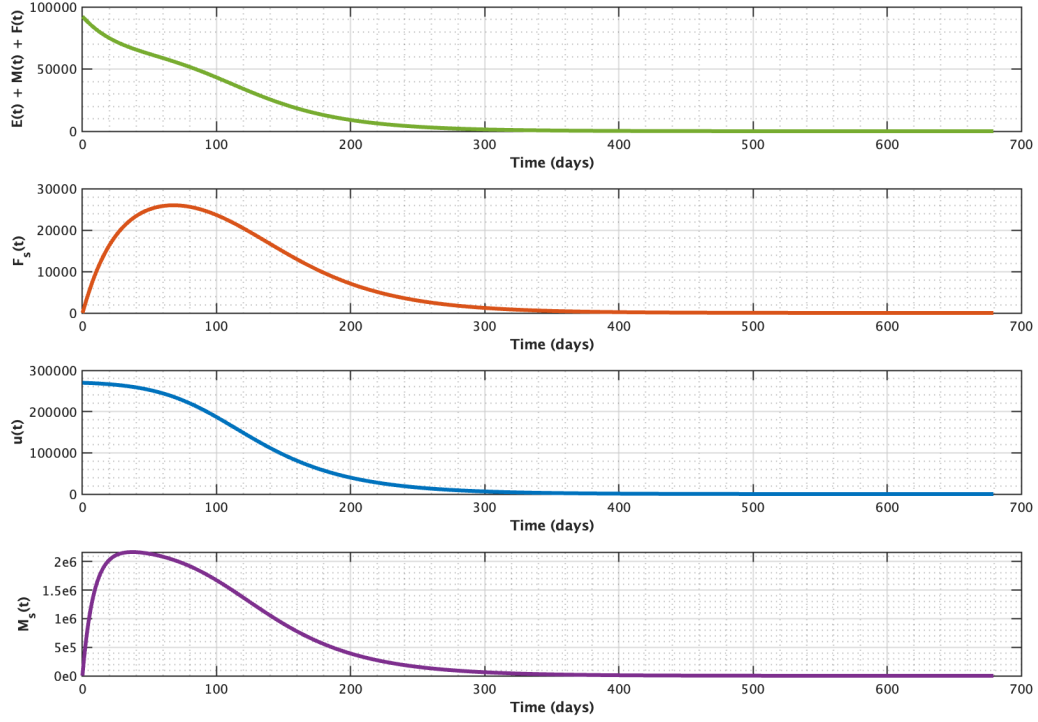


Figure 5.2: Plot of the system (5.4) with the feedback law (5.40) for $\sigma = 2R$, $R = 76.56$.

5.4 Conclusion

The global asymptotic stability of the SIT control system has been proved for many nonlinear feedback control laws (see [1, 2, 46, 29]). In this work, we prove this result for some simple linear laws.

We study linear feedback laws depending on the number of wild male mosquitoes M and egg density E in Section 5.3.2. Thanks to Theorem 5.3.2, for instance, the feedback law $u = 4R(1 - \nu)\nu_E E + 2R(\delta_s - \delta_M)M$ stabilizes the system when females have no preference for fertile males ($\gamma = 1$). In practice, we use ovitraps to measure the egg density in the population. To estimate the wild male density, we use a technique called release-recapture: sterile males are marked before being released into the population. To measure the adult population, specific traps are placed in different areas to capture them. Thanks to their markings, wild males can be distinguished from sterile ones.

We also study linear feedback laws depending on the total number of male mosquitoes, $M + M_s$, and egg density E in Section 5.3.1. Thanks to Theorem 5.3.1, for instance, the feedback law $u = 2R(1 - \nu)\nu_E E + (\delta_s - \delta_M)(M + M_s)$ stabilizes the system to zero under the assumption that females have no preference for fertile males ($\gamma = 1$). The advantage of this linear law is that it depends on the total number of males in the dynamics. It does not require distinguishing wild male mosquitoes from sterile ones in the traps.

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Chapter 6

Feedback stabilization for a spatial-dependent Sterile Insect Technique model with Allee Effect

This chapter is a joint work with Luis Almeida and Jean-Michel Coron.

Abstract. This work focuses on feedback control strategies for applying the sterile insect technique (SIT) to eliminate pest populations. The presentation is centered on the case of mosquito populations, but most of the results can be extended to other species by adapting the model and selecting appropriate parameter values to describe the reproduction and movement dynamics of the species under consideration. In our study, we address the spatial distribution of the population in a two dimensional bounded domain by extending the temporal SIT model analyzed in [2], thereby obtaining a reaction-diffusion SIT model. After the analysis of the existence and the uniqueness of the solution of this problem, we construct a feedback law that globally asymptotically stabilizes the extinction equilibrium thus yielding a robust strategy to keep the pest population at very low levels in the long term.

6.1 Introduction

The Sterile Insect Technique (SIT) involves releasing sterile males of the pest population to disrupt its reproduction [23, 54, 110]. Therefore, this type of intervention affects only the target insect species, offering a significant environmental advantage compared to the use of pesticides, which are often harmful to many species, including humans, and against which pests can quickly develop resistance. The presentation is centered in the case of mosquito populations, but most of the results can be extended to other species by adapting the model and selecting appropriate parameter values to describe the reproduction and movement dynamics of the species under consideration.

In this paper we consider a non-empty regular open subset Ω of $\mathbb{R}^2$ and Neumann boundary conditions (which correspond to having a zero flux of mosquitoes through the boundary of the domain). To describe the spatial dispersion of the adult mosquitoes, we employ a system of reaction-diffusion equations to model the dynamics of the mosquito population. Biologically, this mathematical model is suitable for representing, for instance, a population of mosquitoes established within an island ecosystem or any other domain that is isolated from the external mosquito populations (in the zero flux sense). It is also possible to consider more complex boundary conditions (see, for instance, [11]).

The primary focus of this work is the well-posedness of such system in L^1 and its control using the release of sterile males in the setting of SIT.

The SIT model as an ODE system has been extensively studied from various mathematical perspectives, including the construction of feedback laws (discrete, periodic, and continuous release) relying on optimal control [12, 32], monotone systems [33, 29], Lyapunov functions [2, 1], and Deep Reinforcement Learning (DRL) [4, 3]. To propose alternative strategies for releasing sterile males, spatial dynamics have also been investigated using PDE models to describe the spatial invasion of mosquitoes. Notably, various control strategies have been developed to address this problem, such as the “wave blocking” control strategies presented in [13], the “corridor/barrier

strategy” introduced in [17], and the “rolling carpet” control strategies described in [8, 75]. In the latter, the techniques are designed to block or push back the invasion of disease vectors or pests.

In the present contribution, we also cover the situation where the mosquitoes are already well established in the ecosystem. In this setting, we construct a feedback control law to stabilize the spatio-temporal SIT model around the extinction equilibrium. This work improves and extends our previous results on the ODE model [2, 1].

The outline of the paper is as follows. In Section 6.2, we study the ODE SIT model with an Allee effect term of the form $\eta M/(1 + \eta M)$ which allows us to take into account the probability that an emerging female mosquito finds a male to mate. In Section 6.3, we consider the well-posedness of a reaction-diffusion model for the mosquito population life cycle on smooth, bounded, open sets. In Section 6.4, we construct a feedback control law for the SIT reaction-diffusion model and analyze the well-posedness of the resulting closed-loop system.

6.2 ODE Model for Mosquito Population with Allee Effect

We denote by $\beta_E > 0$ the oviposition rate, $\delta_E, \delta_F, \delta_M > 0$ the death rates for eggs, females, and wild adult males, respectively, $\nu_E > 0$ the hatching rate for eggs, $\nu \in (0, 1)$ the probability that a pupa gives rise to a female ($1 - \nu$ is the probability of giving rise to a male). Here, E represents the mosquito density in the aquatic phase (which, in this work, with a slight abuse of language, we will often designate by eggs to simplify), F is the density of adult females, M is the wild adult male density, and K is the carrying capacity for the aquatic phase. To take into account the difficulty of the females to find a male with whom to mate (in order to produce fertilized eggs) when the density of the population is low, we introduce an Allee effect given by a Michaelis-Menten type factor $\frac{\eta M}{1 + \eta M}$ where η is the search efficiency parameter: $1/\eta$ corresponding the constant male density for which the mating probability of a female would be $1/2$ (see [49] where the notation is $\theta = 1/\eta$). As described in [49] (where the previous form is called the rectangular hyperbola function), other forms can be used for representing this Allee effect but, for simplicity, we chose to use this basic one in this work. We will discuss this issue and the probabilistic motivations of the different forms in a future work. Most of the results of this work will be valid for similar expressions like, for instance, the slightly more complicated exponential form described in [49] as could be expected from their very similar behavior (as seen in figures 1 and 2 of [49]).

The native population dynamics with Allee effect will be given by

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) \frac{\eta M}{1 + \eta M} - (\nu_E + \delta_E)E, \quad (6.1)$$

$$\dot{F} = \nu \nu_E E - \delta_F F, \quad (6.2)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M. \quad (6.3)$$

For the sake of readability, most of the proofs of the statements in this section are postponed to Appendix 6.7. Let us define:

$$r := 1 + \frac{2\delta_M}{\eta K(1 - \nu)\nu_E} \left(1 + \sqrt{1 + \frac{\eta K(1 - \nu)\nu_E}{\delta_M}}\right), \quad (6.4)$$

$$\tilde{r} := 1 + \frac{2\delta_M}{\eta K(1 - \nu)\nu_E} \left(1 - \sqrt{1 + \frac{\eta K(1 - \nu)\nu_E}{\delta_M}}\right), \quad (6.5)$$

$$R := \frac{\beta_E \nu \nu_E}{\delta_F(\nu_E + \delta_E)}, \quad (6.6)$$

$$E_0 := \frac{K}{2} \left(1 - \frac{1}{R}\right), \quad (6.7)$$

$$E_1 = \frac{K}{2R} \left(R - 1 - \sqrt{(R - \tilde{r})(R - r)}\right) \text{ if } R > r, \quad (6.8)$$

$$E_2 = \frac{K}{2R} \left(R - 1 + \sqrt{(R - \tilde{r})(R - r)}\right) \text{ if } R > r, \quad (6.9)$$

and, for $E \in \mathbb{R}$,

$$X_E := \left(E, \frac{\nu \nu_E}{\delta_F} E, \frac{(1 - \nu)\nu_E}{\delta_M} E\right)^T. \quad (6.10)$$

The next theorem, whose proof is given in Appendix 6.7, gives the steady states of (6.1)-(6.2)-(6.3).

Theorem 6.2.1.

- (i) If $R \in (0, r)$, the system (6.1)-(6.2)-(6.3) has a unique steady state, called the extinction equilibrium, $\mathbf{0} = (0, 0, 0)^T$.
- (ii) If $R = r$, besides the extinction equilibrium, the system (6.1)-(6.2)-(6.3) has a unique steady state X_{E_0} .
- (iii) If $R \in [r, +\infty)$, besides the extinction equilibrium, the system (6.1)-(6.2)-(6.3) has two equilibria X_{E_1} and X_{E_2} .

Our next theorem, also proved in Appendix 6.7, is dealing with the asymptotic stability of the equilibria given in Theorem 6.2.1 for the system (6.1)-(6.2)-(6.3).

Theorem 6.2.2. For the system (6.1)-(6.2)-(6.3) on $[0, +\infty)^3$,

- (i) For every value of R , $\mathbf{0}$ is locally asymptotically stable;
- (ii) If $R < r$, $\mathbf{0}$ is globally asymptotically stable on $[0, +\infty)^3$;
- (iii) If $R = r$, then

$$X_{E_0} = (E_0, F_0, M_0)^T \text{ is global asymptotically stable on } \left\{ (E, F, M)^T \in [0, +\infty)^3 : E_0 \leq E \leq K, F_0 \leq F \text{ and } M_0 \leq M \right\}, \quad (6.11)$$

$\mathbf{0}$ is globally asymptotically stable on

$$\left\{ (E, F, M)^T \in [0, +\infty)^3 : E < E_0, F < F_0 \text{ and } M < M_0 \right\}, \quad (6.12)$$

$$X_{E_0} \text{ is unstable.} \quad (6.13)$$

- (iv) If $R > r$, X_{E_2} is locally asymptotically stable, while X_{E_1} is unstable.

The proof of (ii), which is given in Appendix 6.7, relies on the monotonicity property of the dynamical system (6.1)-(6.2)-(6.3). However, when $R < 1$, the global asymptotic stability of $\mathbf{0}$ can also be proven using Lyapunov's second theorem. To this end, we define a candidate Lyapunov function $V : [0, +\infty)^3 \rightarrow [0, +\infty)$, $p = (E, F, M)^T \mapsto V(p)$, as in the proof of [2, Theorem 2.2]. It is given by:

$$V(p) := \frac{1+R}{1-R}E + \frac{2\beta_E}{\delta_F(1-R)}F + M. \quad (6.14)$$

Since $R < 1$, it follows that:

$$V(p) > V(\mathbf{0}) = 0, \quad \forall p \in [0, +\infty)^3 \setminus \{\mathbf{0}\}, \quad (6.15)$$

$$V(p) \rightarrow +\infty \quad \text{as } |p| \rightarrow +\infty \quad \text{with } p \in [0, +\infty)^3. \quad (6.16)$$

Moreover, along the trajectories of (6.1)-(6.2)-(6.3), we have:

$$\dot{V}(p) = -(\nu\nu_E + \delta_E)E - \frac{\beta_E}{K} \frac{1+R}{1-R} FE \frac{\eta M}{1+\eta M} - \delta_M M - \beta_E F - \frac{1+R}{1-R} \frac{1}{1+\eta M} \beta_E F, \quad (6.17)$$

which leads to:

$$\dot{V}(p) \leq -(\nu\nu_E + \delta_E)E - \delta_M M - \beta_E F - \delta_s M_s. \quad (6.18)$$

From (6.14) and (6.18), we deduce:

$$\dot{V}(p) \leq -c_0 V(p), \quad (6.19)$$

where

$$c_0 := \min \left\{ \frac{(\nu\nu_E + \delta_E)(1-R)}{1+R}, \frac{\delta_F(1-R)}{2}, \delta_M \right\} > 0. \quad (6.20)$$

Remark 6.2.1. The above proof shows the global exponential stability and gives an estimate of the exponential decay rate. It would be interesting to provide a Lyapunov function for the case $R \in [1, r)$.

Let us denote by $M_s(t) \geq 0$ the sterile adult male density, $\delta_s > 0$ the death rate of sterile adults, $u \geq 0$ the control which is the density of sterile males released at time t , and $0 < \gamma \leq 1$ accounts for the fact that females may have a preference for fertile males. Conversely, $\gamma > 1$ would correspond to females showing a preference for sterile males. This situation can occur in certain species for instance when sterile males are enhanced with pheromones or a high-protein diet. Such a scenario is even more favorable for the success of SIT and thus we will concentrate on the harder setting of having $0 < \gamma \leq 1$.

When we release sterile male mosquitoes with a release function u , we obtain the following dynamics

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) \frac{\eta M}{1 + \eta(M + \gamma M_s)} - (\nu_E + \delta_E)E, \quad (6.21)$$

$$\dot{F} = \nu \nu_E E - \delta_F F, \quad (6.22)$$

$$\dot{M} = (1 - \nu) \nu_E E - \delta_M M, \quad (6.23)$$

$$\dot{M}_s = u - \delta_s M_s. \quad (6.24)$$

The fact that we had not considered the Allee effect in our previous work [2] made that the model had a singularity when M and M_s converged to zero, which is no longer the case in the present model. Removing this singularity by taking into account the mating difficulties in small populations also has the advantage of simplifying many of the proofs thanks to being able to apply more directly the classical results in control theory in smooth settings.

If $u = 0$, the eigenvalues of the Jacobian of system (6.21)-(6.23)-(6.22)-(6.24) at $\mathbf{0} \in \mathbb{R}^4$ are $-(\nu_E + \delta_E)$, $-\delta_F$, $-\delta_M$, and $-\delta_s$. They are all real and negative, which implies that $\mathbf{0}$ is locally asymptotically stable for the system (6.21)-(6.23)-(6.22)-(6.24). If $u = 0$, the steady states are $(E, F, M, 0)^T$, where $(E, F, M)^T$ are steady states of (6.1)-(6.2)-(6.3) and one has the following theorem, whose proof is also given in Appendix 6.7.

Theorem 6.2.3. *If $R < r$, then $\mathbf{0}$ is globally asymptotically stable on $[0, +\infty)^4$ for the system (6.21)-(6.22)-(6.23)-(6.24) with $u = 0$.*

6.3 Reaction diffusion model for mosquito populations

Here, we study the effect of releasing sterile males in a limited region in space among a mosquito population already established in a non-empty open subset Ω of $\mathbb{R}^2$. For the sake of simplicity, we will assume that Ω is regular enough and that all biological parameters remain constant over time, thus disregarding the effects of field heterogeneity and seasonal variations, except for the carrying capacity K .

Few studies explicitly model the spatial component due to the lack of sufficient knowledge about vectors in the field. Moreover, from a mathematical perspective, the study of spatial-temporal models is more sophisticated. A reaction-diffusion equation was used in [81] to model the spread of a pest in a SIT model. A significant amount of research has been conducted in the simplified setting of traveling wave equations, leading to elaborate control strategies (see, for instance, [13, 8, 75]).

In this work, we do not consider the one-dimensional traveling wave simplification. Instead, we study the life cycle model of a mosquito population on a domain Ω , which is a non-empty regular open subset of $\mathbb{R}^2$.

We choose the space $L^1(\Omega)$ as our working space. In the context of mathematical biology, the space L^1 (the space of absolutely integrable functions) offers several advantages that make it particularly useful and often more relevant than L^2 and H^1 , which are typically chosen due to their convenience for mathematical analysis. Indeed, the L^1 -norm measure quantities such as total biomass, total population, or total resource consumption in biological models. This direct interpretability makes L^1 spaces especially relevant in applications where the integral of a function over its domain Ω has a clear biological meaning like for a density function of a population in which case its L^1 norm yields the total population of the corresponding species in the domain. In this section, we will study the existence and uniqueness of the solution in such a working space.

6.3.1 Life cycle model of a mosquito population in a the bounded domain

We assume that the carrying capacity function $K \in C^0(\bar{\Omega}; (0, +\infty))$. Let $(E^0, F^0, M^0)^T : \Omega \rightarrow [0, +\infty)^3$ be the mosquito population at the initial time $t = 0$. The mosquito population density

$$(E(t, x), F(t, x), M(t, x))^T$$

at time $t \geq 0$ and position $x \in \Omega$ is the solution of the Cauchy problem

$$\begin{cases} \frac{\partial E}{\partial t} = \beta_E F \left(1 - \frac{E}{K(x)}\right) \frac{\eta M}{1 + \eta M} - (\nu_E + \delta_E)E, & t \geq 0, x \in \Omega, \\ \frac{\partial F}{\partial t} - d_1 \Delta F = \nu \nu_E E - \delta_F F, & t \geq 0, x \in \Omega, \\ \frac{\partial M}{\partial t} - d_2 \Delta M = (1 - \nu) \nu_E E - \delta_M M, & t \geq 0, x \in \Omega, \\ \frac{\partial F}{\partial n} = \frac{\partial M}{\partial n} = 0, & t \geq 0, x \in \partial\Omega, \\ (E(0, x), F(0, x), M(0, x))^T = (E^0(x), F^0(x), M^0(x))^T, & x \in \Omega, \end{cases} \quad (6.25)$$

where n is the outward unit normal to Ω .

A first natural mathematical question is the well-posedness of this Cauchy problem, i.e. the existence and the uniqueness of solution for (6.25) in a meaningful class. We study this question in the next section, i.e. in Section 6.3.2. In Section 6.3.3 we study the global asymptotic stability of the origin for (6.25).

6.3.2 Well-posedness of the Cauchy problem

Let us start by recalling the motivation of the definition of weak solutions of the Cauchy problem (6.25). Let $T > 0$, $(E, F, M)^T : [0, T] \times \bar{\Omega} \rightarrow [0, +\infty)^3$ be a smooth solution of (6.25) on $[0, T] \times \bar{\Omega}$, and let $\varphi : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ be of class C^1 . Let $t \in [0, T]$. Multiplying the second and third equations of (6.25), integrating over $[0, t] \times \Omega$, performing integration by parts, and using the last two equations of (6.25), one gets

$$\begin{aligned} \int_{\Omega} F(t, x) \varphi(t, x) dx - \int_{\Omega} F^0(x) \varphi(0, x) dx + d_1 \int_{Q_t} \nabla F \cdot \nabla \varphi - \int_{Q_t} F \frac{\partial \varphi}{\partial t} = \\ \int_{Q_t} (\nu \nu_E E - \delta_F F) \varphi, \end{aligned} \quad (6.26)$$

$$\begin{aligned} \int_{\Omega} M(t, x) \varphi(t, x) dx - \int_{\Omega} M^0(x) \varphi(0, x) dx + d_2 \int_{Q_t} \nabla M \cdot \nabla \varphi - \int_{Q_t} M \frac{\partial \varphi}{\partial t} = \\ \int_{Q_t} ((1 - \nu) \nu_E E - \delta_M M) \varphi, \end{aligned} \quad (6.27)$$

with

$$Q_t := (0, t) \times \Omega. \quad (6.28)$$

Conversely, if $(E, F, M)^T : [0, T] \times \bar{\Omega} \rightarrow [0, +\infty)^3$ is smooth enough (for example of class C^2) and is such that

$$\begin{aligned} \frac{\partial E}{\partial t}(t, x) = \beta_E F(t, x) \left(1 - \frac{E(t, x)}{K(x)}\right) \frac{\eta M(t, x)}{1 + \eta M(t, x)} \\ - (\nu_E + \delta_E) E(t, x), \quad \forall t \in [0, T], \text{ for almost every } x \in \Omega, \end{aligned} \quad (6.29)$$

$$E(0, x) = E^0(x) \text{ for almost every } x \in \Omega, \quad (6.30)$$

and such that (6.26) and (6.27) hold for every $t \in [0, T]$ and for every $\varphi : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ of class C^1 , then $(E, F, M)^T$ is a solution of the Cauchy problem (6.25) on $[0, T]$. Hence it is reasonable to define the notion of weak solution so that (6.26) and (6.27) hold. It is then natural to adopt the following definition.

Definition 6.3.1. *Let $T > 0$. The application $(E, F, M)^T : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}^3$ is a weak solution of the Cauchy problem (6.25) on $[0, T]$ if*

$$E, F, \text{ and } M \text{ take values in } [0, +\infty), \quad (6.31)$$

$$E, F, \text{ and } M \text{ are in } C^0([0, T]; L^1(\Omega)), \quad (6.32)$$

$$F \text{ and } M \text{ are in } L^1((0, T); W^{1,1}(\Omega)), \quad (6.33)$$

and (6.29), (6.30) hold, and (6.26) and (6.27) hold for every $t \in [0, T]$ and for every $\varphi : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ of class C^1 . Moreover $(E, F, M)^T : [0, +\infty) \times \Omega \rightarrow \mathbb{R}^3$ is a weak solution of the Cauchy problem (6.25) on $[0, +\infty)$ if, for every $T > 0$, its restriction to $[0, T] \times \Omega$ is a weak solution of (6.25) on $[0, T]$.

Remark 6.3.1. *Let us comment about the meaning of (6.29). Let $x \in \Omega$ be such that*

$$F(\cdot, x) \text{ and } M(\cdot, x) \text{ are in } L^1(0, T). \quad (6.34)$$

Note that, since F and M are in $C^0([0, T]; L^1(\Omega)) \subset L^1((0, T) \times \Omega)$, (6.34) holds for almost every $x \in \Omega$. For x such that (6.34) holds, $t \in [0, T] \mapsto E(t, x)$ is just a classical solution of the time-varying ordinary differential equation

$$\dot{\psi} = A(t, x) - B(t, x) \psi \quad (6.35)$$

with

$$A(t, x) := \beta_E F(t, x) \frac{\eta M(t, x)}{1 + \eta M(t, x)} \text{ and } B(t, x) = \frac{\beta_E}{K(x)} F(t, x) \frac{\eta M(t, x)}{1 + \eta M(t, x)} + (\nu_E + \delta_E). \quad (6.36)$$

In particular, if moreover $E(0, x) = E^0(x)$, which by (6.30) holds also for almost every $x \in \Omega$,

$$E(t, x) = e^{-\int_0^t B(s, x) ds} E^0(x) + \int_0^t e^{-\int_s^t B(\tau, x) d\tau} A(s, x) ds. \quad (6.37)$$

With this definition, one has the following well-posedness theorem, whose proof is given in Appendix 6.8 and in Appendix 6.10.

Theorem 6.3.1. *Let $(E^0, F^0, M^0)^T : \Omega \rightarrow [0, +\infty)^3$ be such that*

$$E^0 \in L^1(\Omega), F^0 \in L^1(\Omega), \text{ and } M^0 \in L^1(\Omega). \quad (6.38)$$

Then there exists a weak solutions of the Cauchy problem (6.25) on $[0, +\infty)$. Moreover, if one also has

$$E^0 \in L^r(\Omega) \text{ for some } r \in (1, +\infty], \quad (6.39)$$

this weak solution is unique.

Remark 6.3.2. *It would be interesting to know if the uniqueness of the weak solution also holds without assuming (6.39).*

6.3.3 Asymptotic stability of the life cycle model

Our next theorem shows that if (6.40) holds then the origin is globally exponentially stable for (6.25) in the $L^1(\Omega)^3$ -norm.

Theorem 6.3.2. *Assume that*

$$R < 1. \quad (6.40)$$

Then there exist $C > 0$ and $\mu > 0$ such that, for every $(E^0, F^0, M^0)^T : \Omega \rightarrow [0, +\infty)^3$ such that (6.38) holds, every weak solution $(t, x) \in [0, +\infty) \times \Omega \mapsto (E(t, x), F(t, x), M(t, x))^T$ of (6.25) satisfies

$$\|E(t, \cdot)\|_{L^1(\Omega)} + \|F(t, \cdot)\|_{L^1(\Omega)} + \|M(t, \cdot)\|_{L^1(\Omega)} \leq C e^{-\mu t} (\|E^0\|_{L^1(\Omega)} + \|F^0\|_{L^1(\Omega)} + \|M^0\|_{L^1(\Omega)}) \quad \forall t \geq 0. \quad (6.41)$$

Proof. Let us now define $L : L^1(\Omega)^3 \rightarrow \mathbb{R}^3$ by

$$L((E, F, M)^T) := \frac{1+R}{1-R} \int_{\Omega} E + \frac{2\beta_E}{\delta_F(1-R)} \int_{\Omega} F + \int_{\Omega} M. \quad (6.42)$$

(Compare with (6.14).) Note that, for every $(E, F, M) \in L^1(\Omega)^3$ satisfying

$$E \geq 0, F \geq 0, \text{ and } M \geq 0 \text{ almost everywhere in } \Omega, \quad (6.43)$$

one has

$$a (\|E\|_{L^1(\Omega)} + \|F\|_{L^1(\Omega)} + \|M\|_{L^1(\Omega)}) \leq L((E, F, M)^T) \leq A (\|E\|_{L^1(\Omega)} + \|F\|_{L^1(\Omega)} + \|M\|_{L^1(\Omega)}) \quad (6.44)$$

with

$$a := \min \left\{ \frac{1+R}{1-R}, \frac{2\beta_E}{\delta_F(1-R)}, 1 \right\} > 0, \quad (6.45)$$

$$A := \max \left\{ \frac{1+R}{1-R}, \frac{2\beta_E}{\delta_F(1-R)}, 1 \right\} > 0. \quad (6.46)$$

Let $(E^0, F^0, M^0)^T : \Omega \rightarrow [0, +\infty)^3$ be such that (6.38) holds and let

$$(t, x) \in [0, +\infty) \times \Omega \mapsto (E(t, x), F(t, x), M(t, x))^T$$

be a weak solution of (6.25). With a slight abuse of notation, let us define $L : [0, +\infty) \rightarrow [0, +\infty)$ by

$$L(t) := L((E(t, \cdot), F(t, \cdot), M(t, \cdot))^T), \quad (6.47)$$

where V is defined in (6.14). From (6.29) one has, in the sense of distribution on $(0, T)$,

$$\frac{\partial E}{\partial t} \leq \beta_E F - (\nu_E + \delta_E)E \text{ in } L^1((0, T) \times \Omega), \quad (6.48)$$

which implies that, in the sense of distribution on $(0, T)$,

$$\frac{d}{dt} \int_{\Omega} E(t, x) dx \leq \int_{\Omega} (\beta_E F - (\nu_E + \delta_E)E), \quad \forall t \in [0, +\infty) \quad (6.49)$$

Taking $\varphi \equiv 1$ in (6.26) and (6.27), we get for every $t \in [0, +\infty)$,

$$\int_{\Omega} F(t, x) dx - \int_{\Omega} F^0(x) dx = \int_{Q_t} (\nu \nu_E E - \delta_F F), \quad (6.50)$$

$$\int_{\Omega} M(t, x) dx - \int_{\Omega} M^0(x) dx = \int_{Q_t} ((1 - \nu) \nu_E E - \delta_M M). \quad (6.51)$$

Using (6.49) and taking the derivative with respect to t in (6.50) and (6.51), and using (6.47), we get, in the sense of distributions, on $(0, +\infty)$,

$$\dot{L}(t) \leq -(\nu \nu_E + \delta_E) \int_{\Omega} E(t) - \beta_E \int_{\Omega} F(t) - \delta_M \int_{\Omega} M(t), \quad (6.52)$$

In (6.52) and in the following, we use the usual convention $E(t)(x) = E(t, x)$, $F(t)(x) = F(t, x)$, and $M(t)(x) = M(t, x)$. From (6.42), (6.52), and (6.47), one has

$$\dot{L}(t) \leq -cL(t), \quad (6.53)$$

with

$$c := \min \left\{ \frac{(\nu \nu_E + \delta_E)(1 - R)}{1 + R}, \frac{\delta_F(1 - R)}{2}, \delta_M \right\} > 0, \quad (6.54)$$

which, using (6.44), (6.45), and (6.46), concludes the proof of Theorem 6.3.2. $\square$

6.4 Reaction diffusion model for SIT

The SIT model presented here is derived from an extinction of the temporal model (6.25) by adding the sterile male mosquitoes. The study is conducted in a regular bounded open set $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \frac{\partial E}{\partial t} = \beta_E F \left(1 - \frac{E}{K(x)} \right) \frac{\eta M}{1 + \eta(M + \gamma M_s)} - (\nu_E + \delta_E)E, & t \geq 0, x \in \Omega, \\ \frac{\partial F}{\partial t} - d_1 \Delta F = \nu \nu_E E - \delta_F F, & t \geq 0, x \in \Omega, \\ \frac{\partial M}{\partial t} - d_2 \Delta M = (1 - \nu) \nu_E E - \delta_M M, & t \geq 0, x \in \Omega, \\ \frac{\partial M_s}{\partial t} - d_3 \Delta M_s = u - \delta_s M_s, & t \geq 0, x \in \Omega, \\ \frac{\partial F}{\partial n} = \frac{\partial M}{\partial n} = \frac{\partial M_s}{\partial n} = 0, & t \geq 0, x \in \partial\Omega, \end{cases} \quad (6.55)$$

where $M_s(t, x) \geq 0$ represents the sterile adult male density, $\delta_s > 0$ is the death rate of sterile adults, $u \geq 0$ is the control representing the density of sterile males released at each time and, as before, γ is a positive constant representing the preference of females for sterile or fertile males. The probability that a female mates with a fertile male is given by $\frac{\eta M}{1 + \eta(M + \gamma M_s)}$.

6.4.1 Well-posedness of the closed-loop system

Concerning the feedback law $u : [0, +\infty)^4 \rightarrow \mathbb{R}$, $(E, F, M, M_s)^T \mapsto u(y)$, we always assume the existence of $C_u > 0$ such that

$$0 \leq u((E, F, M, M_s)^T) \leq C_u(1 + E + F + M + M_s), \quad \forall (E, F, M, M_s)^T \in [0, +\infty)^4. \quad (6.56)$$

Moreover, for the uniqueness of the weak solution, we consider the following condition:

$$\left\{ \begin{array}{l} \text{For every } \mathcal{E} > 0, \text{ there exists } C_u(\mathcal{E}) > 0 \text{ such that, for every} \\ (E, F, M, M_s, \hat{E}, \hat{F}, \hat{M}, \hat{M}_s)^T \in [0, +\infty)^8 \text{ satisfying } E + \hat{E} \leq \mathcal{E}, \\ |u((E, F, M, M_s)^T) - u((\hat{E}, \hat{F}, \hat{M}, \hat{M}_s)^T)| \leq \\ C_u(\mathcal{E}) \left(|E - \hat{E}| + |F - \hat{F}| + (1 + F + \hat{F}) (|M - \hat{M}| + |M_s - \hat{M}_s|) \right). \end{array} \right. \quad (6.57)$$

We are interested in the Cauchy problem for the closed-loop system, i.e.

$$\left\{ \begin{array}{l} \frac{\partial E}{\partial t} = \beta_E F \left(1 - \frac{E}{K(x)}\right) \frac{\eta M}{1 + \eta(M + \gamma M_s)} - (\nu_E + \delta_E)E, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial F}{\partial t} - d_1 \Delta F = \nu \nu_E E - \delta_F F, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial M}{\partial t} - d_2 \Delta M = (1 - \nu) \nu_E E - \delta_M M, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial M_s}{\partial t} - d_3 \Delta M_s = u((E, F, M, M_s)^T) - \delta_s M_s, \quad t \geq 0, \quad x \in \Omega, \\ \frac{\partial F}{\partial n} = \frac{\partial M}{\partial n} = \frac{\partial M_s}{\partial n} = 0, \quad t \geq 0, \quad x \in \partial\Omega, \\ (E(0, x), F(0, x), M(0, x), M_s(0, x))^T = (E^0(x), F^0(x), M^0(x), M_s^0(x))^T, \quad x \in \Omega. \end{array} \right. \quad (6.58)$$

Following the same heuristic as for Definition 6.3.1, we adopt the following definition.

Definition 6.4.1. Let $T > 0$. The application $(E, F, M, M_s)^T : [0, T] \times \Omega \rightarrow \mathbb{R}^4$ is a weak solution of the Cauchy problem (6.58) on $[0, T]$ if

$$E, F, M \text{ and } M_s \text{ take values in } [0, +\infty), \quad (6.59)$$

$$E, F, M, \text{ and } M_s \text{ are in } C^0([0, T]; L^1(\Omega)), \quad (6.60)$$

$$F, M, \text{ and } M_s \text{ are in } L^1((0, T); W^{1,1}(\Omega)), \quad (6.61)$$

$$\begin{aligned} \frac{\partial E}{\partial t}(t, x) &= \beta_E F(t, x) \left(1 - \frac{E(t, x)}{K(x)}\right) \frac{\eta M(t, x)}{1 + \eta(M(t, x) + \gamma M_s(t, x))} \\ &\quad - (\nu_E + \delta_E)E(t, x) \quad t \in (0, T], \text{ for almost every } x \in \Omega, \end{aligned} \quad (6.62)$$

$$E(0, x) = E^0(x) \text{ for almost every } x \in \Omega, \quad (6.63)$$

and, for every $t \in [0, T]$ and for every $\varphi : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ of class C^1 , one has (6.26), (6.27), and

$$\begin{aligned} \int_{\Omega} M_s(t, x) \varphi(t, x) \, dx - \int_{\Omega} M_s^0(x) \varphi(0, x) \, dx + d_3 \int_{Q_t} \nabla M_s \cdot \nabla \varphi - \int_{Q_t} M_s \frac{\partial \varphi}{\partial t} = \\ \int_{Q_t} \left(u((E, F, M, M_s)^T) - \delta_s M_s \right) \varphi. \end{aligned} \quad (6.64)$$

Moreover $(E, F, M, M_s)^T : [0, +\infty) \times \Omega \rightarrow \mathbb{R}^4$ is a weak solution of the Cauchy problem (6.58) on $[0, +\infty)$ if, for every $T > 0$, its restriction to $[0, T] \times \Omega$ is a weak solution of (6.58) on $[0, T]$.

(For the meaning of (6.62), see Remark 6.3.1 above). With this definition, one has the following well-posedness theorem, whose proof is given in Appendix 6.9 and in Appendix 6.10.

Theorem 6.4.1. Let $(E^0, F^0, M^0, M_s^0)^T : \Omega \rightarrow [0, +\infty)^4$ be such that

$$E^0 \in L^1(\Omega), \quad F^0 \in L^1(\Omega), \quad M^0 \in L^1(\Omega), \text{ and } M_s^0 \in L^1(\Omega). \quad (6.65)$$

Then, there exists a weak solution of the Cauchy problem (6.58) on $[0, +\infty)$. Moreover, if one also assume that

(6.57) holds and that

$$E^0 \in L^\infty(\Omega) \text{ and } F^0 \in L^r(\Omega) \text{ for some } r > 1, \quad (6.66)$$

then this weak solution is unique.

Remark 6.4.1. Once more, it would be interesting to know if the uniqueness of the weak solution also holds without assuming (6.66).

6.4.2 Design of state-feedback controllers

In this section, we construct a control law to stabilize the mosquito population at zero in Ω . The construction process is based on the backstepping method for stabilizing ODE control systems. This approach was previously applied to the ODE SIT model in a prior study [1]; see also [2, 46] for other ODE SIT models. Let $\theta \in [0, +\infty)$. We assume that, for every $(t, x) \in [0, T] \times \Omega$, $M_s(t, x) = \theta M(t, x)$. This leads to the consideration of the Cauchy problem

$$\begin{cases} \frac{\partial E}{\partial t} = \beta_E F \left(1 - \frac{E}{K(x)}\right) \frac{\eta M}{1 + \eta(1 + \gamma\theta)M} - (\nu_E + \delta_E)E, & t \geq 0, x \in \Omega, \\ \frac{\partial F}{\partial t} - d_1 \Delta F = \nu \nu_E E - \delta_F F, & t \geq 0, x \in \Omega, \\ \frac{\partial M}{\partial t} - d_2 \Delta M = (1 - \nu)\nu_E E - \delta_M M, & t \geq 0, x \in \Omega, \\ \frac{\partial F}{\partial n} = \frac{\partial M}{\partial n} = 0, & t \geq 0, x \in \partial\Omega, \\ (E(0, x), F(0, x), M(0, x))^T = (E^0(x), F^0(x), M^0(x))^T, & x \in \Omega, \end{cases} \quad (6.67)$$

Of course, Theorem 6.3.1 dealing with the Cauchy problem (6.3.1) also holds for the Cauchy problem (6.67). Let us define

$$\mathcal{R}(\theta) := \frac{R}{1 + \gamma\theta}. \quad (6.68)$$

Then, one has the following proposition.

Proposition 6.4.1. Assume that

$$\mathcal{R}(\theta) < 1. \quad (6.69)$$

Then there exists $C > 0$ and $\mu > 0$ such that, for every $(E^0, F^0, M^0)^T : \Omega \rightarrow [0, +\infty)^3$ such that (6.38) holds, every weak solution $(t, x) \in [0, +\infty) \times \Omega \mapsto (E(t, x), F(t, x), M(t, x))^T$ of (6.67) satisfies

$$\|E(t, \cdot)\|_{L^1(\Omega)} + \|F(t, \cdot)\|_{L^1(\Omega)} + \|M(t, \cdot)\|_{L^1(\Omega)} \leq C e^{-\mu t} (\|E^0\|_{L^1(\Omega)} + \|F^0\|_{L^1(\Omega)} + \|M^0\|_{L^1(\Omega)}) \quad \forall t \geq 0. \quad (6.70)$$

Proof. The proof of this proposition is essentially the same as the proof of Theorem 6.3.2. Let us now define $L : L^1(\Omega)^3 \rightarrow \mathbb{R}^3$ by (compare with (6.42) and (6.14))

$$L((E, F, M)^T) := \frac{1 + \mathcal{R}(\theta)}{1 - \mathcal{R}(\theta)} \int_{\Omega} E + \frac{2\beta_E}{\delta_F(1 - \mathcal{R}(\theta))} \int_{\Omega} F + \int_{\Omega} M. \quad (6.71)$$

Note that, for every $(E, F, M) \in L^1(\Omega)^3$ satisfying (6.43), one still has (6.44) but now with

$$a := \min \left\{ \frac{1 + \mathcal{R}(\theta)}{1 - \mathcal{R}(\theta)}, \frac{2\beta_E}{\delta_F(1 - \mathcal{R}(\theta))}, 1 \right\} > 0, \quad (6.72)$$

$$A := \max \left\{ \frac{1 + \mathcal{R}(\theta)}{1 - \mathcal{R}(\theta)}, \frac{2\beta_E}{\delta_F(1 - \mathcal{R}(\theta))}, 1 \right\} > 0. \quad (6.73)$$

One still gets (6.53), but now with c defined by (compare with (6.54))

$$c := \min \left\{ \frac{(\nu \nu_E + \delta_E)(1 - \mathcal{R}(\theta))}{1 + \mathcal{R}(\theta)}, \frac{\delta_F(1 - \mathcal{R}(\theta))}{2}, \delta_M \right\} > 0. \quad (6.74)$$

As in the proof of Theorem 6.3.2, this concludes the proof of Proposition 6.4.1. $\square$

Let us now move on to the stabilization of the system (6.55). As we will see, the proof of Proposition 6.4.1, together with the backstepping method (see, for example, [44, pages 334–335]), leads us to consider the feedback law $y = (x^T, M_s)^T \mapsto u((x^T, M_s)^T) \in [0, +\infty)$, where $u : [0, +\infty)^4 \rightarrow [0, +\infty)$ is defined by

$$u(y) := F \frac{\psi M(\theta M + M_s)^2}{\alpha(M + \gamma M_s + \kappa)(\kappa + (1 + \gamma\theta)M)(3\theta M + M_s)} + \theta(\delta_s - \delta_M)M \frac{(\theta M + 3M_s)}{(3\theta M + M_s)}, \quad (6.75)$$

where

$$\delta_s > \delta_M, \quad (6.76)$$

$$\kappa := \frac{1}{\eta}, \quad (6.77)$$

$$\psi := \frac{\gamma\beta_E(1 + \mathcal{R}(\theta))}{1 - \mathcal{R}(\theta)}. \quad (6.78)$$

Even if the feedback law (6.75) is not of class C^1 , it satisfies both (6.56) and (6.57). In particular Theorem 6.4.1 can be applied for this feedback law.

With the feedback (6.75), one has the following global stability result.

Theorem 6.4.2. *Assume that $d_2 = d_3$. Let us choose θ large enough so that (6.69) holds and let us choose $\alpha > 0$ small enough so that*

$$1 - \frac{3\alpha\theta(1 - \nu)\nu_E}{2(\nu\nu_E + \delta_E)} > 0. \quad (6.79)$$

Then there exists $C > 0$ and $\mu > 0$ such that, for every $(E^0, F^0, M^0, M_s^0)^T : \Omega \rightarrow [0, +\infty)^4$ such that (6.65) and (6.66) hold

$$\|E(t, \cdot)\|_{L^1(\Omega)} + \|F(t, \cdot)\|_{L^1(\Omega)} + \|M(t, \cdot)\|_{L^1(\Omega)} + \|M_s(t, \cdot)\|_{L^1(\Omega)} \leqslant C e^{-\mu t} (\|E^0\|_{L^1(\Omega)} + \|F^0\|_{L^1(\Omega)} + \|M^0\|_{L^1(\Omega)} + \|M_s^0\|_{L^1(\Omega)}) \quad \forall t \geqslant 0, \quad (6.80)$$

where $(t, x) \in [0, +\infty) \times \Omega \mapsto (E(t, x), F(t, x), M(t, x), M_s(t, x))^T \in [0, +\infty)^4$ is the weak solution of the closed-loop system (6.58) with the feedback law (6.75) satisfying the initial condition

$$E(0, x) = E^0(x), \quad F(0, x) = F^0(x), \quad M(0, x) = M^0(x), \quad M_s(0, x) = M_s^0(x) \quad \forall x \in \Omega. \quad (6.81)$$

Remark 6.4.2. *It would be interesting to know if in this theorem one could remove the assumption (6.66) and get (6.80) for every weak solution of the closed-loop system (6.58) with the feedback law (6.75) satisfying the initial condition (6.81). Note that if we remove the assumption (6.66), we have the existence of a weak solution but we do not know if this solution is unique.*

Proof. Let $V : [0, +\infty)^3 \rightarrow [0, +\infty)$ be defined by (compare with (6.71))

$$V((E, F, M)^T) = \frac{1 + \mathcal{R}(\theta)}{1 - \mathcal{R}(\theta)} E + \frac{2\beta_E}{\delta_F(1 - \mathcal{R}(\theta))} F + M. \quad (6.82)$$

Following the backstepping method, we penalize the inequality $M_s \neq \theta M$ by considering the Lyapunov function $U : L^1(\Omega; [0, +\infty))^4 \rightarrow [0, +\infty)$, $(E, F, M, M_s)^T \mapsto U((E, F, M, M_s)^T)$ defined by

$$U := \int_{\Omega} V + \alpha \int_{\Omega} \frac{(\theta M - M_s)^2}{\theta M + M_s}, \quad (6.83)$$

with the convention

$$\frac{(\theta M - M_s)^2}{\theta M + M_s} = 0 \text{ if } M = M_s = 0. \quad (6.84)$$

Simple computations show that, for every $(M, M_s)^T \in [0, +\infty)^2$ such that $M + M_s > 0$,

$$\min\{\alpha, \frac{1}{1 + \theta}\}(M + M_s) \leqslant M + \alpha \frac{(\theta M - M_s)^2}{\theta M + M_s} \leqslant \max\{\alpha, (1 + \alpha\theta)\}(M + M_s), \quad (6.85)$$

which leads to

$$k'_1 \|y\|_{L^1(\Omega)^4} \leqslant U \leqslant k'_2 \|y\|_{L^1(\Omega)^4}, \quad \forall y \in L^1(\Omega; [0, +\infty))^4, \quad (6.86)$$

where

$$k'_1 := \min\left\{\frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)}, \alpha, \frac{1}{1+\theta}, \frac{2\beta_E}{\delta_F(1-\mathcal{R}(\theta))}\right\}, \quad (6.87)$$

$$k'_2 := \max\left\{\frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)}, \alpha, 1+\alpha\theta, \frac{2\beta_E}{\delta_F(1-\mathcal{R}(\theta))}\right\}. \quad (6.88)$$

Let us define $W : [0, +\infty)^4 \rightarrow [0, +\infty)$ by

$$W(y) := V(y) + \alpha \frac{(\theta M - M_s)^2}{\theta M + M_s}. \quad (6.89)$$

Let $v := (E, M, F)^T$, so that $y = (v^T, M_s)^T$. With a slight abuse of notation, we define $V(y) := V(v)$. Let

$$H(y) := \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) \frac{\eta M}{1+\eta(1+\gamma\theta)M} - (\nu_E + \delta_E)E \\ (1-\nu)\nu_E E - \delta_M M \\ \nu\nu_E E - \delta_F F \\ u(y) - \delta_s M_s \end{pmatrix}. \quad (6.90)$$

Let $T > 0$. Let us consider, for the moment, trajectories $t \in [0, T] \rightarrow [0, +\infty)^4$ of (6.58) such that

$$y_i \in H^{1,2}(Q_T) \quad \forall i \in \{2, 3, 4\}, \quad (6.91)$$

where

$$H^{1,2}(Q_T) := \left\{ \phi : Q_T \rightarrow \mathbb{R} : \phi, \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial^2 \phi}{\partial x_1^2}, \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \text{ and } \frac{\partial^2 \phi}{\partial x_2^2} \text{ are all in } L^2(Q_T) \right\}. \quad (6.92)$$

Along these trajectories of (6.58), using once more Stokes' theorem,

$$\frac{dU}{dt} = \int_{\Omega} \left(\nabla V(y) \cdot H(y) + \alpha \nabla \left(\frac{(\theta M - M_s)^2}{\theta M + M_s} \right) \cdot H(y) \right) - \int_{\Omega} \sum_{i=2}^4 d_{i-1} \nabla y_i \cdot \nabla \left(\frac{\partial W}{\partial y_i} \right). \quad (6.93)$$

Note that

$$\begin{aligned} \int_{\Omega} \nabla V(y) \cdot H(y) &= \int_{\Omega} \nabla V(y) \cdot H((v^T, \theta M)^T) \\ &\quad + \int_{\Omega} \nabla V(y) \cdot (H((v^T, M_s)^T) - H((v^T, \theta M)^T)), \end{aligned} \quad (6.94)$$

and that, from (6.82) and (6.90),

$$\begin{aligned} \nabla V(y) \cdot (H((v^T, M_s)^T) - H((v^T, \theta M)^T)) &= \\ &= \frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)} \beta_E F \left(1 - \frac{E}{K}\right) \frac{\gamma M(\theta M - M_s)}{(M + \gamma M_s + \kappa)(\kappa + (1 + \gamma\theta)M)}. \end{aligned} \quad (6.95)$$

Hence, using also (6.78), we have

$$\nabla V(y) \cdot H(y) = \nabla V(v) \cdot h(v) + F \left(1 - \frac{E}{K}\right) \frac{\psi M(\theta M - M_s)}{(M + \gamma M_s + \kappa)(\kappa + (1 + \gamma\theta)M)}. \quad (6.96)$$

Moreover, for the second term in (6.93), we have

$$\begin{aligned} \alpha \nabla \left(\frac{(\theta M - M_s)^2}{\theta M + M_s} \right) \cdot H(y) &= \alpha \frac{\theta M - M_s}{(\theta M + M_s)^2} \left(\theta((1-\nu)\nu_E E - \delta_M M)(\theta M + 3M_s) \right. \\ &\quad \left. - u(y)(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s) \right) \end{aligned} \quad (6.97)$$

From (6.17), (6.93) and (6.97), we get

$$\begin{aligned} \frac{dU(t)}{dt} \leq & -\beta_E \int_{\Omega} F - \delta_M \int_{\Omega} M - \frac{1 + \mathcal{R}(\theta)}{1 - \mathcal{R}(\theta)} \frac{\beta_E}{K} \int_{\Omega} \frac{MFE}{(1 + \gamma\theta)M + \kappa} - (\nu\nu_E + \delta_E) \int_{\Omega} E \\ & + \alpha\theta(1 - \nu)\nu_E \int_{\Omega} \frac{\theta M(\theta M + 3M_s)}{(\theta M + M_s)^2} E \\ & + \int_{\Omega} \alpha \frac{\theta M - M_s}{(\theta M + M_s)^2} \left(F \left(1 - \frac{E}{K} \right) \frac{\psi M(\theta M + M_s)^2}{\alpha(M + \gamma M_s + \kappa)(\kappa + (1 + \gamma\theta)M)} - \theta \delta_M M(\theta M + 3M_s) \right. \\ & \left. - u(y)(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s) \right) - \int_{\Omega} \sum_{i=2}^4 d_{i-1} \nabla y_i \cdot \nabla \left(\frac{\partial W}{\partial y_i} \right). \end{aligned} \quad (6.98)$$

Note that

$$\frac{\theta M(\theta M + 3M_s)}{(\theta M + M_s)^2} \leq 3 \quad \text{in } \Omega \quad (6.99)$$

and that

$$\begin{aligned} -(\nu\nu_E + \delta_E) \int_{\Omega} E + \alpha\theta(1 - \nu)\nu_E \int_{\Omega} \frac{M(\theta M + 3M_s)}{(\theta M + M_s)^2} E \leq \\ -(\nu\nu_E + \delta_E) \left(1 - \frac{3\alpha\theta(1 - \nu)\nu_E}{(\nu\nu_E + \delta_E)} \right) \int_{\Omega} E \end{aligned} \quad (6.100)$$

Let

$$\sigma := 1 - \frac{3\alpha\theta(1 - \nu)\nu_E}{(\nu\nu_E + \delta_E)}. \quad (6.101)$$

Note that (6.79) implies that

$$\sigma > 0. \quad (6.102)$$

From (6.98), (6.99), (6.100), (6.101), and (6.102), one gets

$$\begin{aligned} \frac{dU(t)}{dt} \leq & -\beta_E \int_{\Omega} F - \delta_M \int_{\Omega} M - (\nu\nu_E + \delta_E)\sigma \int_{\Omega} E - \alpha(1 - \nu)\nu_E \int_{\Omega} \frac{M_s(\theta M + 3M_s)}{(\theta M + M_s)^2} E \\ & + \int_{\Omega} \alpha \frac{\theta M - M_s}{(\theta M + M_s)^2} \left(F \left(1 - \frac{E}{K} \right) \frac{\psi M(\theta M + M_s)^2}{\alpha(M + \gamma M_s + \kappa)(\kappa + (1 + \gamma\theta)M)} - \theta \delta_M M(\theta M + 3M_s) \right. \\ & \left. - u(y)(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s) \right) - \int_{\Omega} \sum_{i=2}^4 d_{i-1} \nabla y_i \cdot \nabla \left(\frac{\partial W}{\partial y_i} \right). \end{aligned} \quad (6.103)$$

From (6.89), we have

$$\begin{aligned} \sum_{i=2}^4 d_{i-1} \nabla y_i \cdot \nabla \left(\frac{\partial W}{\partial y_i} \right) &= d_2 \nabla M \cdot \nabla \left(\frac{\partial W}{\partial M} \right) + d_3 \nabla M_s \cdot \nabla \left(\frac{\partial W}{\partial M_s} \right) \\ &= d_2 \frac{8\theta^2 M_s^2}{(\theta M + M_s)^3} |\nabla M|^2 + d_3 \frac{8\theta^2 M^2}{(\theta M + M_s)^3} |\nabla M_s|^2 \\ &\quad - (d_2 + d_3) \frac{8\theta^2 M M_s}{(\theta M + M_s)^3} \nabla M \cdot \nabla M_s. \end{aligned}$$

For $d_2 = d_3 = d$, this gives

$$\begin{aligned} \sum_{i=2}^4 d_{i-1} \nabla y_i \cdot \nabla \left(\frac{\partial W}{\partial y_i} \right) &= d \frac{8\theta^2 M_s^2}{(\theta M + M_s)^3} |\nabla M|^2 + d \frac{8\theta^2 M^2}{(\theta M + M_s)^3} |\nabla M_s|^2 \\ &\quad - 2d \frac{8\theta^2 M M_s}{(\theta M + M_s)^3} \nabla M \cdot \nabla M_s \\ &\geq 0. \end{aligned} \quad (6.104)$$

which, together with (6.75), (6.82), and (6.103), implies the existence of a constant $c > 0$ such that

$$\frac{dU(t)}{dt} \leq -c \int_{\Omega} V(y) + \int_{\Omega} \alpha \frac{\theta M - M_s}{(\theta M + M_s)^2} \left(-\theta \delta_s M(\theta M + 3M_s) + \delta_s M_s(3\theta M + M_s) \right). \quad (6.105)$$

Note that

$$-\theta\delta_s M(\theta M + 3M_s) + \delta_s M_s(3\theta M + M_s) = -\delta_s(\theta M - M_s)(\theta M + M_s). \quad (6.106)$$

So

$$\frac{dU(t)}{dt} \leq -c \int_{\Omega} V(y) - \delta_s \alpha \int_{\Omega} \frac{(\theta M - M_s)^2}{(\theta M + M_s)}, \quad (6.107)$$

$$\leq -c_b U(t), \quad (6.108)$$

where

$$c_b := \min\{c, \delta_s\}. \quad (6.109)$$

Hence

$$\frac{dU}{dt} \leq -c_b U(t), \quad (6.110)$$

which implies that

$$U(t) \leq U(0)e^{-c_b t} \quad \forall t \in [0, T], \quad (6.111)$$

which, together with (6.86) gives (6.80) with

$$\mu = c_b \text{ and } C = \frac{k'_2}{k'_1}. \quad (6.112)$$

It remains to remove assumption (6.91). Note that this assumption is satisfied if $(F^0, M^0, M_s^0)^T$ are smooth enough. For example this is the case if

$$F^0, M^0, M_s^0 \text{ are in } L^\infty(\Omega). \quad (6.113)$$

If (6.113) does not hold, we consider a sequence $(F_n^0, M_n^0, M_{sn}^0)_{n \in \mathbb{N}}^T$ such that

$$(F_n^0, M_n^0, M_{sn}^0)^T \in L^\infty(\Omega)^3 \quad \forall n \in \mathbb{N}, \quad (6.114)$$

$$F_n^0 \geq 0, M_n^0 \geq 0 \text{ and } M_{sn}^0 \geq 0 \quad \forall n \in \mathbb{N}, \quad (6.115)$$

$$F_n^0 \rightarrow F^0 \text{ in } L^1(\Omega), M_n^0 \rightarrow M^0 \text{ in } L^1(\Omega), \text{ and } M_{sn}^0 \rightarrow M_s^0 \text{ in } L^1(\Omega) \text{ as } n \rightarrow +\infty. \quad (6.116)$$

Let y_n be the trajectory of the closed-loop system, i.e. (6.58) with the feedback law (6.75), for the initial data $(E^0, F_n^0, M_n^0, M_{sn}^0)^T$ and let y be the trajectory of the closed-loop system for the initial data $(E^0, F^0, M^0, M_s^0)^T$. One has the following convergence result, whose proof is given in Appendix 6.10 (see (6.264), (6.265), (6.266), (6.288), and (6.289)).

$$\lim_{n \rightarrow +\infty} \|y_n - y\|_{L^1((0,T) \times \Omega)} = 0. \quad (6.117)$$

Let, with a slight abuse of notation,

$$U_n(t) := U(y_n(t)) \text{ and } U(t) := U(y(t)). \quad (6.118)$$

From (6.82), (6.83), (6.117), and (6.118), we have

$$\lim_{n \rightarrow +\infty} \|U_n - U\|_{L^1(0,T)} = 0. \quad (6.119)$$

Property (6.111) for the trajectory y_n is

$$U_n(t) \leq U_n(0)e^{-c_b t} \quad \forall t \in [0, T] \quad \forall n \in \mathbb{N}. \quad (6.120)$$

Letting $n \rightarrow +\infty$ in (6.120) and using (6.116) together with (6.119), relation (6.120) also holds for the y trajectory. Hence, again, (6.80) holds with C and μ defined in (6.112). This concludes the proof of Theorem 6.4.2. $\square$

6.5 Numerical simulation in 2D

This section presents some numerical simulations of system (6.58) to illustrate our analytical results. Since we consider a two-dimensional model, the full discretization is achieved using a second-order finite difference method for spatial discretization and a first-order non-standard finite difference method for temporal discretization. The time step follows a CFL condition to ensure the positivity of the solution.

We consider the domain $\Omega = [0, \ell] \times [0, \ell]$ (where $\ell = 5$ km) with a heterogeneous environmental capacity $K : \Omega \rightarrow (0, +\infty)$. (Note that such a Ω is not smooth; however it is a plane convex polygon, which is sufficient to perform our proofs as it can be seen from the study of the elliptic case, which is for instance done in [55].)

The parameters we use are presented in the following table.

	Parameter name	Typical interval	Value in our work ¹	Unit
β_E	Effective fecundity	[7.46, 14.85]	8	Day ⁻¹
ν_E	Hatching parameter	[0.005, 0.25]	0.05	Day ⁻¹
δ_E	Aquatic phase death rate	[0.023, 0.046]	0.03	Day ⁻¹
δ_F	Female death rate	[0.033, 0.046]	0.04	Day ⁻¹
δ_M	Males death rate	[0.077, 0.139]	0.1	Day ⁻¹
δ_s	Sterilized male death rate	-	0.12	Day ⁻¹
ν	Probability of emergence	-	0.49	
η	Search efficiency parameter	-	0.7	km ²

Table 6.1: Value for the parameters of system (6.25) (see [18][102]).

We always take the diffusion coefficients for wild females and males as $d_1 = d_2 = 0.1$. We choose $\gamma = 1$.

$$K(x, y) := \zeta + \Lambda_1 e^{-\frac{(x-\mu_1)^2 + (y-\xi_1)^2}{\sigma_1}} + \Lambda_2 e^{-\frac{(x-\mu_2)^2 + (y-\xi_2)^2}{\sigma_2}} + \Lambda_3 e^{-\frac{(x-\mu_3)^2 + (y-\xi_3)^2}{\sigma_3}} \quad (6.121)$$

This type of function is appropriate for representing a carrying capacity K , the constant ζ accounting for the unknown egg-laying sites throughout the area. The three Gaussian functions represent the well-known egg-laying sites (water puddles, ponds, etc.). The parameters $\Lambda_i, \sigma_i, \mu_i, \xi_i$ for $i = 1, \dots, 3$ are used to qualify the position and size of the egg-laying sites.

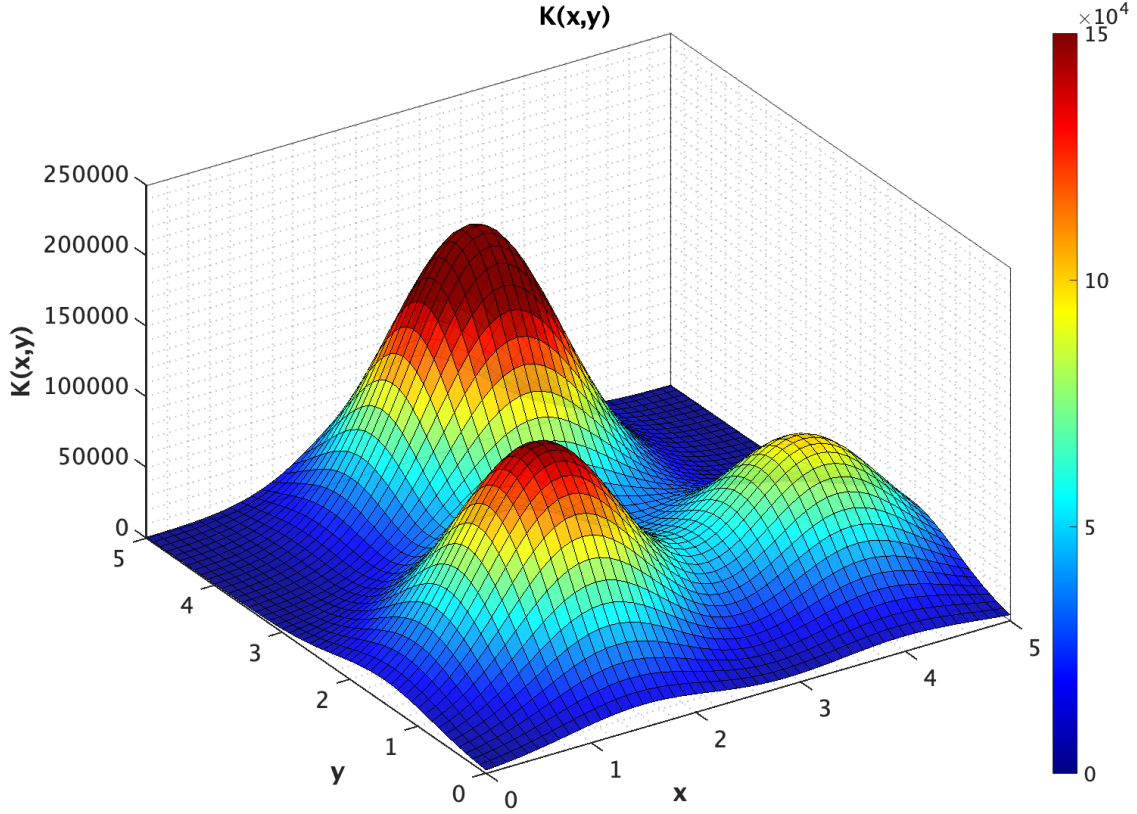


Figure 6.1: Numerical simulation of the function K for $\zeta = 500$, $\Lambda_1 = 2 \times 10^5$, $\Lambda_2 = 1.5 \times 10^5$, $\Lambda_3 = 1 \times 10^5$. The domain $\Omega = [0, \ell] \times [0, \ell]$ and $\ell = 5\text{km}$ which we discretize with $N_x = 50 = N_y$ and $dx = dy = 0.1$. $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $\mu_1 = 2.5$, $\mu_2 = 1.5 = \xi_2 = \xi_3$, $\mu_3 = 4 = \xi_1$.

Using the values chosen in Table 6.1, the L^1 -norm of the carrying capacity K is

$$\|K\|_{L^1} = \int_{\Omega} K(x, y) dx dy = 1.33 \times 10^6. \quad (6.122)$$

The initial condition is the steady state of the system without diffusion:

$$E^0(x, y) = \left(1 - \frac{1}{R}\right) K(x, y), \quad (6.123)$$

$$F^0(x, y) = \frac{\nu \nu_E}{\delta_F} E^0(x, y), \quad (6.124)$$

$$M^0(x, y) = \frac{(1 - \nu) \nu_E}{\delta_M} E^0(x, y). \quad (6.125)$$

Although this simulation does not intend to be a reproduction of real field data, for the quantities of mosquitoes to have the right order of magnitude, we chose the average male density to be the same as the one reported in [54], i.e. we take $(\|E^0\|_{L^1}, \|M^0\|_{L^1}, \|F^0\|_{L^1}) = (1.30 \times 10^6, 3.3 \times 10^5, 8.03 \times 10^5)$.

We apply the control u given by relation (6.75) on the whole domain Ω . We assume that using adult mosquito traps we can have an estimate of the states and use it to compute the feedback law. The following figure presents the result of our simulation. We set $\theta = 75$ and $\alpha = 0.25$.

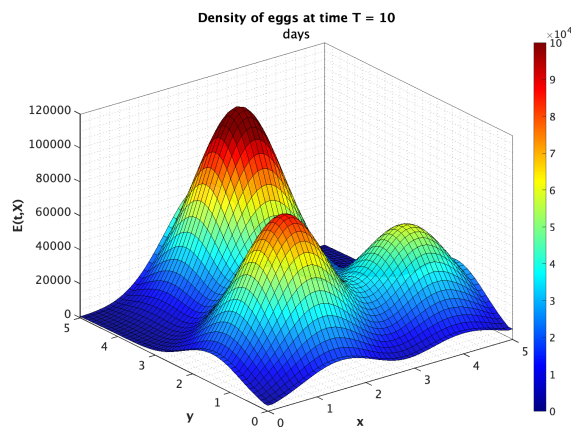


Figure 6.2: Fecundated egg density at the time $t = 10$ days when applying the backstepping feedback law (6.75).

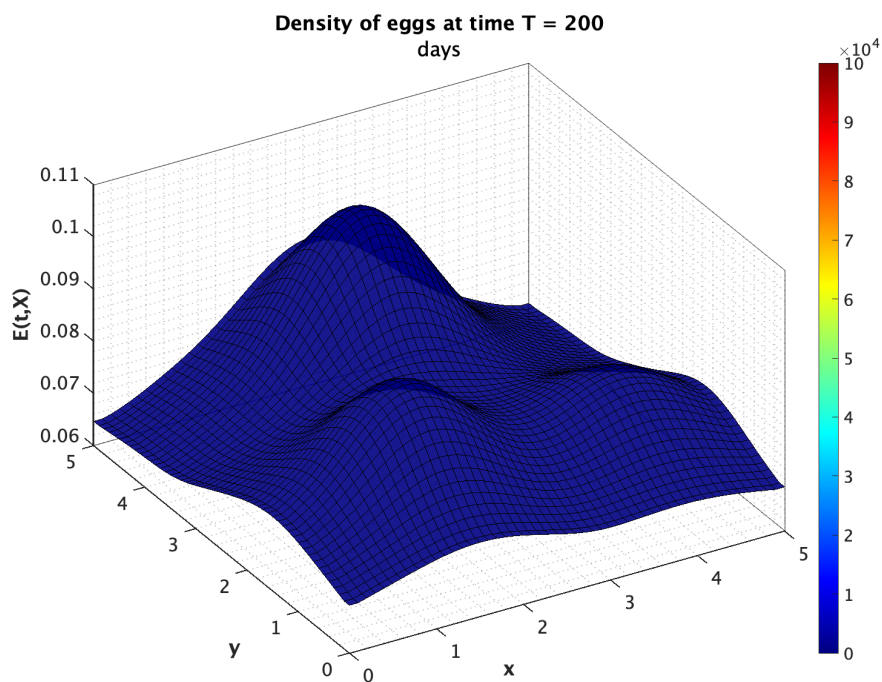


Figure 6.3: Fecundated egg density at the time $t = 200$ days when applying the backstepping feedback law (6.75). We remark that it is very close to zero everywhere in the domain as also shown in Figure 6.5

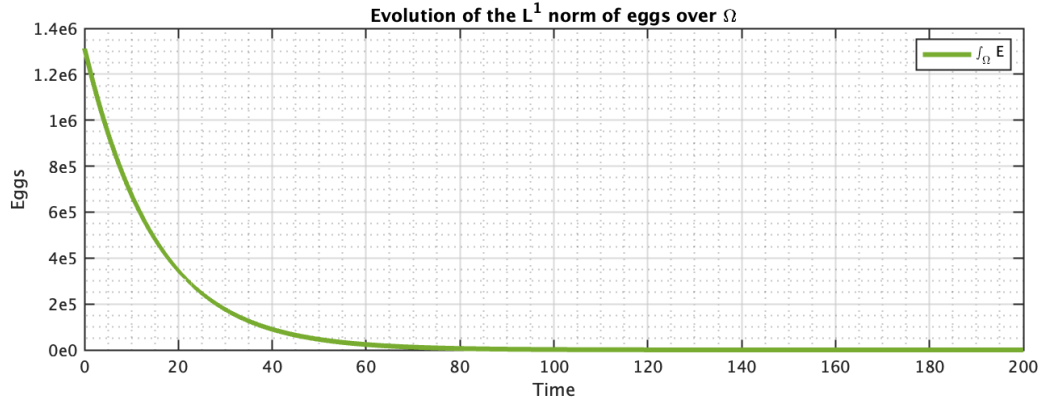


Figure 6.4: Evolution of $t \mapsto \int_{\Omega} E(t)$, representing the total number of fertile eggs across the entire domain.

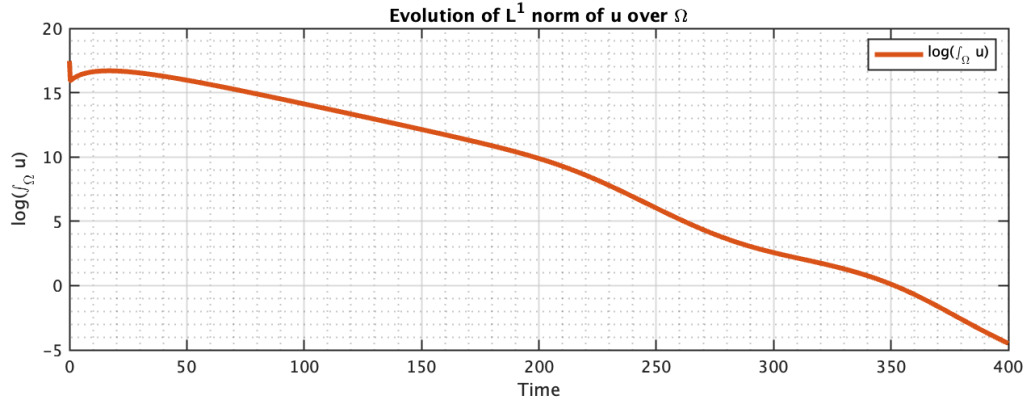


Figure 6.5: Evolution of $t \mapsto \ln \left(\int_{\Omega} u(t) \right)$, representing the total number of sterile mosquitoes released across the entire domain.

Figure 6.3 shows that after time $t = 200$ days, the population density across the entire domain approaches zero. For the control law (6.75), we proved (Theorem 6.4.2) the asymptotic stability of the population when the diffusion coefficients are equal for male and released sterile male mosquitoes. However, the numerical simulations (see Table 6.2) seem to show that this assumption should not be necessary. In particular, the global asymptotic stability seems to hold when the diffusion coefficient of sterile mosquitoes is set to $d_3 = 0$ and $d_2 \neq 0$.

Figure 6.5 displays the evolution of the L^1 norm of the control (6.75) over time. In Table 6.2, the control cost is given by the expression

$$\int_0^T \int_{\Omega} u(t, x) dx dt. \quad (6.126)$$

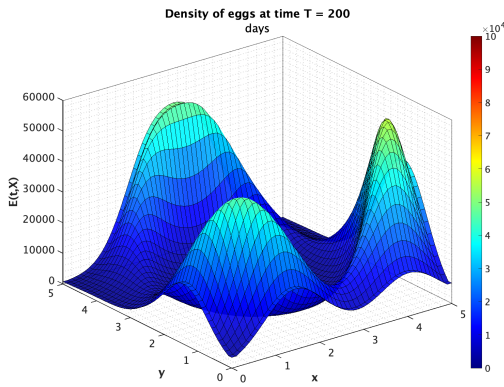
We define the convergence time as the first time $t > 0$ such that

$$\max_{i,j} E(i, j, t) \leq 1. \quad (6.127)$$

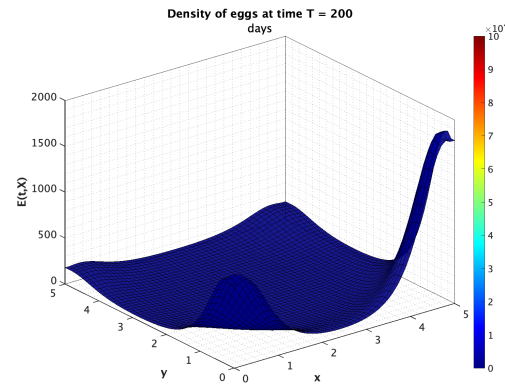
diffusion coefficient	Convergence time	Control cost	Regulation parameter
$d_3 = 0.05$	T = 357 days	1.68×10^8	$\theta = 75, \alpha = 0.25$
$d_3 = 0.05$	T = 167 days	9.4×10^8	$\theta = 75, \alpha = 0.025$
$d_3 = 0$	T = 355 days	1.62×10^8	$\theta = 75, \alpha = 0.25$

Table 6.2: Numerical results obtained for the case where the state feedback law is applied in all Ω .

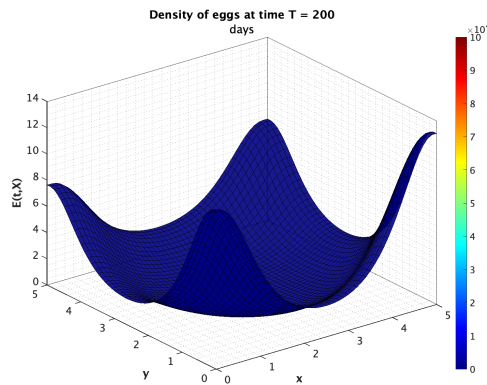
In our proof of Theorem 6.4.2, the control is applied everywhere in Ω . However, it is also interesting to consider, at least numerically, the case where the control is applied only in subset $\omega \subset \Omega$. Knowing that there is a permanent exchange of population between the controlled area ω and the uncontrolled area $\Omega \setminus \omega$ due to the natural spread of mosquitoes, is it possible to stabilize the population in the controlled area at zero? Moreover, by acting only on the subset ω , is it possible to reduce the overall population to zero across the entire domain Ω ? We consider the square centered at $A = (2.5, 2.5)$, $\omega = \{X = (x, y) \in \mathbb{R} : \|X - A\|_\infty \leq \varrho\}$ where $\varrho = 2.5$. We apply the feedback control law (6.75) for $\theta = 75$ and $\alpha = 0.0025$ only in the domain ω . The result is shown in Figures 6.6 and 6.7.



(a) Evolution of the egg density in Ω for $d_3 = 0.01$ at time $T = 200$ days when applying the feedback law (6.75) for $\theta = 75$ and $\alpha = 0.001$.



(b) Evolution of the eggs density in Ω for $d_3 = 0.02$ at time $T = 200$ days when applying the feedback law (6.75) for $\theta = 75$ and $\alpha = 0.001$.



(c) Evolution of the egg density in Ω for $d_3 = 0.03$ at time $T = 200$ days when applying the feedback law (6.75) for $\theta = 75$ and $\alpha = 0.001$.

Figure 6.6

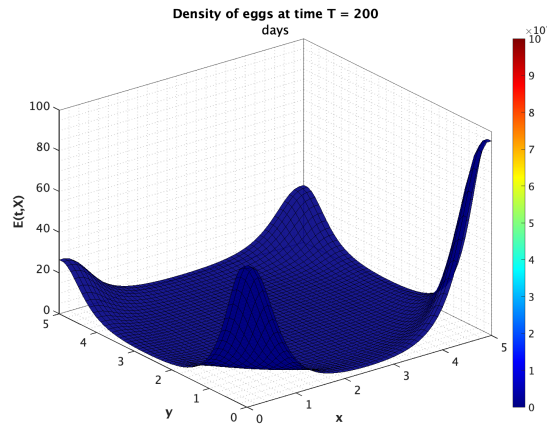


Figure 6.7: Evolution of the egg density in Ω for $d_3 = 0.01$ at time $T = 200$ days when applying the feedback law (6.75) for $\theta = 75$ and $\alpha = 0.0001$.

Figure 6.6 shows that for the same application time $T = 200$ days and the same quantity of sterile males released (feedback control function (6.75) for $\alpha = 0.001$ and $\theta = 75$), the density of eggs converges to zero in the subset ω and in the entire domain Ω , faster when the diffusion coefficient increases. In Figure 6.7, compared to Figure 6.6(a), when we increase the release by choosing $\alpha = 0.0001$ and $\theta = 75$, keeping the other coefficients, the population in the whole domain Ω converges to zero. These figures suggest that the quality of the sterile males, particularly their mobility (which may be reduced by the sterilization and the release processes), can change the rate of convergence in the full domain.

6.6 Conclusion

Very few studies have been done on the spatiotemporal SIT model in more than one-dimensional fields. We first study the ODE case to retrieve for the new model a result concerning the critical points as we had obtained in [2]. It should also be possible to extend the feedback control results of [2, 1] to this new setting (the proofs should even be simpler since we no longer have a singularity at the origin). Next, in section 6.3, we consider a mosquito population mathematical model with diffusion on a domain Ω , which is a non-empty regular open subset of $\mathbb{R}^2$. We have shown the existence and uniqueness of the solution in weak spaces. In particular two components of the state, namely F and M , are assumed to be only L^1 . We favored this Banach space because the L^1 norm is a natural way to represent the total population. We also prove the global asymptotic stability of $\mathbf{0}$ in L^1 if R , defined in (6.6), is less than 1.

In Section 6.4, we study the reaction-diffusion SIT model by showing its well-posedness in weak spaces. In particular, two components of the state, namely M and M_s , are assumed to be only in L^1 . As our objective is to stabilize the mosquito population at a low level across the entire domain, we begin by constructing a feedback law, (6.75), which stabilizes this model throughout Ω . This state feedback law (6.75) is applied across the whole domain Ω . We successfully proved this theoretical result under the condition that the diffusion coefficients are identical for male mosquitoes and released sterile mosquitoes (see Theorem 6.4.2).

Moreover, in Section 6.5, the numerical simulations reveal that stability at the origin is maintained regardless of the diffusion coefficients of sterile mosquitoes when the state feedback law is applied across the entire domain Ω . Nonetheless, the diffusion coefficients should play a more significant role when the control is only implemented in a subdomain of Ω .

We also note that it would be valuable to investigate the case where the diffusion coefficients are not constant in space nor time. Our results can also be extended to higher dimensions. However, even in three dimensions, one would expect that mosquito diffusion differs between horizontal and vertical directions, making it necessary to consider anisotropic diffusion operators.

Acknowledgements

The authors are grateful to Alain Haraux, Thierry Cazenave, Michel Pierre and Michel Souplet for useful information on L^1 solutions for parabolic systems.

6.7 Stability properties of the ODE dynamical systems

6.7.1 Proof of Theorem 6.2.1

By setting the right-hand side of (6.1)-(6.2)-(6.3) to zero, we obtain the extinction equilibrium $\mathbf{0} = (0, 0, 0)^T$ and the non-trivial equilibrium X_E (see (6.10)), where $E \in (0, +\infty)$ satisfies:

$$\frac{\eta\beta_E\nu(1-\nu)\nu_E^2}{\delta_M\delta_F K}E^2 - \frac{(1-\nu)\nu_E(\nu_E + \delta_E)\eta}{\delta_M}\left(\frac{\beta_E\nu\nu_E}{\delta_F(\nu_E + \delta_E)} - 1\right)E + (\nu_E + \delta_E) = 0. \quad (6.128)$$

Using (6.6), we can rewrite (6.128) as

$$\frac{\eta R(1-\nu)\nu_E}{\delta_M K}E^2 - \frac{(1-\nu)\nu_E\eta}{\delta_M}(R-1)E + 1 = 0. \quad (6.129)$$

The discriminant of the quadratic polynomial in E of the left hand side of (6.129) is:

$$\Delta = \frac{\eta(1-\nu)\nu_E}{\delta_M} \left(\frac{\eta(1-\nu)\nu_E}{\delta_M} R^2 - 2\left(\frac{\eta(1-\nu)\nu_E}{\delta_M} + \frac{2}{K}\right)R + \frac{\eta(1-\nu)\nu_E}{\delta_M} \right). \quad (6.130)$$

Let us examine the sign of Δ , which is a quadratic polynomial in R . The discriminant of this polynomial is:

$$\Delta' = \frac{4}{K} \left(\frac{(1-\nu)\nu_E\eta}{\delta_M} + \frac{1}{K} \right) > 0.$$

Thus, the roots of Δ are real and they are r defined in (6.4) and $\tilde{r}$ defined in (6.5). Note that $0 < r, \tilde{r} < r$, and, since $\tilde{r}r = 1$, one has

$$0 < \tilde{r} < 1 < r. \quad (6.131)$$

Let us first study the case $R \in (0, \tilde{r}) \cup (r, +\infty)$. Then there are two solutions of (6.129), which are E_1 defined in (6.8) and E_2 defined in (6.8). Since

$$E_1 E_2 = \frac{\delta_M K}{\eta R(1-\nu)\nu_E} \text{ and } E_1 + E_2 = \frac{K(R-1)}{R}, \quad (6.132)$$

one gets, using also (6.131), that $E_1 < 0$ and $E_2 < 0$ if $R \in (0, \tilde{r})$ (and so these solutions are not relevant), while $E_1 > 0$ and $E_2 > 0$ if $R \in (r, +\infty)$.

Let us now consider the case $R \in \{\tilde{r}, r\}$. Then (6.129) has exactly one solution E_0 defined in (6.7). This solution is in $(0, +\infty)$ if and only if $R > 1$. Hence, using once more (6.131), this solution is in $(0, +\infty)$ if and only if $R = r$.

Finally, let us deal with the case $R \in (\tilde{r}, r)$. Then $\Delta < 0$ and (6.129) has no real solutions.

This concludes the proof of Theorem 6.2.1. $\square$

6.7.2 Proof of Theorem 6.2.2

We define $\xi := (E, F, M)^T$ and $[0, +\infty)^3 := \{\xi = (E, F, M)^T \in \mathbb{R}^3 : E \geq 0, F \geq 0, M \geq 0\}$. The model (6.1)-(6.2)-(6.3) can be expressed in the form

$$\dot{\xi} = f(\xi), \quad (6.133)$$

where $f : [0, +\infty)^3 \rightarrow \mathbb{R}^3$ represents the right-hand side of (6.1)-(6.2)-(6.3):

$$f(\xi) = \begin{pmatrix} f_1(\xi) \\ f_2(\xi) \\ f_3(\xi) \end{pmatrix} = \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) \frac{\eta M}{1+\eta M} - (\nu_E + \delta_E)E \\ \nu\nu_E E - \delta_F F \\ (1-\nu)\nu_E E - \delta_M M \end{pmatrix}. \quad (6.134)$$

The function f is of class $\mathcal{C}^1$ on $[0, +\infty)^3$. Note that if $\dot{\xi} = f(\xi)$ and $\xi(0) \in [0, +\infty)^3$, then, for every $t \geq 0$, $\xi(t)$ exists and belongs to $[0, +\infty)^3$.

Let us first prove (i). The Jacobian matrix of f evaluated at the extinction equilibrium, is given by:

$$J(\mathbf{0}) = \begin{pmatrix} -(\nu_E + \delta_E) & 0 & 0 \\ \nu\nu_E & -\delta_F & 0 \\ (1-\nu)\nu_E & 0 & -\delta_M \end{pmatrix}. \quad (6.135)$$

The eigenvalues of $J(\mathbf{0})$ are $-(\nu_E + \delta_E)$, $-\delta_F$, and $-\delta_M$. All these eigenvalues are real and negative, which implies that the extinction equilibrium $\mathbf{0}$ is locally asymptotically stable for the system (6.133).

Let us now prove (ii). We assume that $R \in (0, r)$ and then, by Theorem 6.2.1,

$$(f(\xi) = \mathbf{0}) \Leftrightarrow (\xi = \mathbf{0}). \quad (6.136)$$

Let us define

$$B := \{\xi \in [0, +\infty)^3 : E \leq K\}. \quad (6.137)$$

Since

$$(E = K) \Rightarrow (f_1(\xi) < 0) \quad (6.138)$$

the set B is positively invariant for $\dot{\xi} = f(\xi)$. Moreover, since

$$(\xi \notin B) \Rightarrow (f_1(\xi) \leq -\delta_E E), \quad (6.139)$$

for every $\xi^0 \in [0, +\infty)^3$ there exists $t \geq 0$ such that $\xi(t) \in B$, where $\xi : [0, +\infty) \rightarrow [0, +\infty)^3$ is the solution of (6.133) satisfying $\xi(0) = \xi^0$. So, in order to prove (ii), it suffices to check that

$$\mathbf{0} \text{ is globally asymptotically stable on } B \text{ for (6.133)}. \quad (6.140)$$

To prove (6.140), we first point out that the system (6.133) is cooperative on B , i.e. f satisfies:

$$\frac{\partial f_i}{\partial x_j}(\xi) \geq 0 \quad \forall \xi \in B, \quad \forall i \in \{1, 2, 3\}, \quad \forall j \in \{1, 2, 3\} \text{ such that } i \neq j. \quad (6.141)$$

We recall the following theorem [18, Theorem 6], which holds for every function f of class C^1 on B such that $\dot{\xi} = f(\xi)$ is cooperative on B .

Theorem 6.7.1. *Let a and b be in B and such that $a_i \leq b_i$ for every $i \in \{1, 2, 3\}$. Assume that $f_i(b) \leq 0 \leq f_i(a)$ for every $i \in \{1, 2, 3\}$. Then $[a, b] := \{\xi \in [0, +\infty)^3 : a_i \leq \xi_i \leq b_i \quad \forall i \in \{1, 2, 3\}\} \subset B$ is positively invariant for $\dot{\xi} = f(\xi)$. Moreover, if $[a, b]$ contains a unique equilibrium p for $\dot{\xi} = f(\xi)$, then p is globally asymptotically stable on $[a, b]$ for $\dot{\xi} = f(\xi)$.*

We apply this theorem with

$$a = (0, 0, 0)^T, \quad (6.142)$$

$$b = b_\lambda = (K, \lambda, \lambda)^T, \quad (6.143)$$

where $\lambda \in (0, +\infty)$. One has

$$\cup_{\lambda \geq \lambda_0} [a, b_\lambda] = B \quad \forall \lambda_0, \quad (6.144)$$

$$\text{there exists } \lambda_0 > 0 \text{ such that } f_i(b_\lambda) \leq 0 \quad \forall i \in \{1, 2, 3\}, \quad \forall \lambda > \lambda_0. \quad (6.145)$$

Property (6.140) readily follows from (6.136), Theorem 6.7.1, (6.144), and (6.145). This concludes the proof of (ii).

Let us now prove (iii). The proof of (6.11) is similar to the proof of (6.140): just replace (6.142) by $a = X_{E_0}$. The instability of X_{E_0} (property (6.13)) readily follows from (6.12) since X_{E_0} belongs to the closure of $\{(E, F, M)^T \in [0, +\infty)^3 : E < E_0, F < F_0 \text{ and } M < M_0\}$. All that remains is to prove (6.12). For this proof, let, for $\varepsilon \in (0, E_0)$,

$$b^\varepsilon := \begin{pmatrix} E_0 - \varepsilon \\ \frac{\nu \nu_E}{\delta_F} (E_0 - \varepsilon) \\ \frac{(1-\nu)\nu_E}{\delta_M} (E_0 - \varepsilon) \end{pmatrix}. \quad (6.146)$$

Clearly

$$f_2(b^\varepsilon) = f_3(b^\varepsilon) = 0 \quad \forall \varepsilon \in (0, E_0). \quad (6.147)$$

Since $R = r$, one gets, using (6.4), (6.6), and (6.7),

$$K = \frac{4R\delta_M}{\eta(1-\nu)\nu_E(R-1)^2}, \quad (6.148)$$

$$E_0 = \frac{K}{2R}(R-1). \quad (6.149)$$

Using (6.134), (6.148), and (6.149), one gets that

$$\frac{f_1(b^\varepsilon)}{\varepsilon^2} = -\rho^2 \frac{\delta_M(\nu_E + \delta_E)(E_0 - \varepsilon)}{\delta_M + (1-\nu)\nu_E\eta(E_0 - \varepsilon)}, \quad (6.150)$$

with

$$\rho := \frac{\eta(1-\nu)\nu_E(R-1)}{2\delta_M}. \quad (6.151)$$

Note that for $R = r$, (6.4), (6.6), and (6.151) imply that

$$\rho > 0, \quad (6.152)$$

which, together with (6.150), implies that

$$\text{Therefore, there exists } \varepsilon_0 \in (0, E_0) \text{ such that } f_1(b^\varepsilon) \leq 0 \ \forall \varepsilon \in (0, \varepsilon_0]. \quad (6.153)$$

Note that, for every $\varepsilon \in (0, E_0)$,

$$(f(\xi) = \mathbf{0} \text{ and } \xi \in [\mathbf{0}, b^\varepsilon]) \Leftrightarrow (\xi = \mathbf{0}), \quad (6.154)$$

$$[\mathbf{0}, b^\varepsilon] \subset B. \quad (6.155)$$

From Theorem 6.7.1, (6.154), and (6.155) one gets that, for every $\varepsilon \in (0, \varepsilon_0]$,

$$\mathbf{0} \text{ is globally asymptotically stable on } [\mathbf{0}, b^\varepsilon] \text{ for } \dot{\xi} = f(\xi). \quad (6.156)$$

Since

$$\cup_{\varepsilon \in (0, \varepsilon_0]} [\mathbf{0}, b^\varepsilon] = \left\{ (E, F, M)^T \in [0, +\infty)^3 : E < E_0, F < F_0 \text{ and } M < M_0 \right\}, \quad (6.157)$$

(6.12) follows from (6.156).

Finally, let us prove (iv). In this case we assume that $R > r$. Let X_E be an equilibrium of f (recall (6.10)). By (6.139),

$$E < K. \quad (6.158)$$

The Jacobian matrix of f evaluated at the equilibrium X_E , is given by:

$$J_f(X_E) = \begin{pmatrix} -\frac{(\nu_E + \delta_E)}{(1 - \frac{E}{K})} & \frac{\beta_E}{R} & \frac{(\nu_E + \delta_E)\delta_M^2}{\eta(1-\nu)^2\nu_E^2 R} \frac{1}{E(1 - \frac{E}{K})} \\ \nu\nu_E & -\delta_F & 0 \\ (1-\nu)\nu_E & 0 & -\delta_M \end{pmatrix} \quad (6.159)$$

Its characteristic polynomial is $P(\lambda) = \lambda^3 + Q_1\lambda^2 + Q_2\lambda + Q_3$, where

$$Q_1 := \delta_M + \delta_F + \frac{(\nu_E + \delta_E)}{1 - \frac{E}{K}}, \quad (6.160)$$

$$Q_2 := (\nu_E + \delta_E) \frac{\eta R(1-\nu)\nu_E(\delta_M E + \delta_F \frac{E^2}{K}) - \delta_M^2}{\eta R(1-\nu)\nu_E E(1 - \frac{E}{K})} + \delta_M \delta_F, \quad (6.161)$$

$$Q_3 := (\nu_E + \delta_E)\delta_M \delta_F \left[\frac{\eta R(1-\nu)\nu_E \frac{E^2}{K} - \delta_M}{\eta R(1-\nu)\nu_E E(1 - \frac{E}{K})} \right]. \quad (6.162)$$

For $E = E_2$, we have

$$Q_1 > 0, \frac{Q_1 Q_2 - Q_3}{Q_1} > 0, \text{ and } Q_3 > 0. \quad (6.163)$$

Thus, by the Routh criterion, all roots of P have strictly negative real parts. Hence, X_{E_2} is locally asymptotically stable for (6.133).

For $E = E_1$ it is easy to prove that $Q_3 < 0$. Therefore X_{E_1} is unstable for (6.133).

6.7.3 Proof of Theorem 6.2.3

In this section, we assume that $u = 0$ and that $R < r$. As in the proof of (ii) of Theorem 6.2.2,

$$B_s := \left\{ (\xi^T, M_s)^T \in [0, +\infty)^4 : E \leq K \right\} \text{ is positively invariant for (6.21)-(6.22)-(6.23)-(6.24),} \quad (6.164)$$

and it suffices to prove that

$$(0, 0, 0, 0)^T \text{ is globally asymptotically stable on } B_s \text{ for (6.21)-(6.22)-(6.23)-(6.24).} \quad (6.165)$$

As $u = 0$, (6.24) implies that, for every $t \geq 0$,

$$M_s(t) = M_s(0)e^{-\delta_s t}. \quad (6.166)$$

Substituting this equation into the system (6.21)-(6.22)-(6.23), we obtain:

$$\dot{\xi} = g(t, \xi), \quad (6.167)$$

where

$$g(t, \xi) := \begin{pmatrix} \beta_E F \left(1 - \frac{E}{K}\right) \frac{\eta M}{1 + \eta M + \eta M_s(0)e^{-\delta_s t}} - (\nu_E + \delta_E)E \\ \nu \nu_E E - \delta_F F \\ (1 - \nu)\nu_E E - \delta_M M \end{pmatrix}. \quad (6.168)$$

Let us point out that

$$g_i(t, \xi) \leq f_i(\xi) \quad \forall \xi \in B, \quad \forall t \geq 0, \quad \forall i \in \{1, 2, 3\}. \quad (6.169)$$

Hence, using also (6.141), the Kamke comparison principle [63] (see also, for example, [43, Theorem 10, Chapter I, Section 4, page 29] or [69, Lemma 4.2, Chapter 1, Section 4, page 51]) gives that

$$\left(\dot{\xi} = f(\xi), \quad \dot{\tilde{\xi}} = g(t, \tilde{\xi}), \quad \text{and } \xi(0) = \tilde{\xi}(0) \in B \right) \Rightarrow \left(\tilde{\xi}_i(t) \leq \xi_i(t) \quad \forall i \in \{1, 2, 3\}, \quad \forall t \geq 0 \right), \quad (6.170)$$

which, together with the proof of (ii) of Theorem 6.2.2 given in Section 6.7.2, implies Theorem 6.2.3.

6.8 Proof of Theorem 6.3.1 when (6.39) holds

In this section we assume that (6.39) holds and prove that the Cauchy problem (6.25) has a unique weak solution on $[0, +\infty)$. For $E \in C^0([0, T]; L^1(\Omega))$, we define

$$\|E\|_{C_T^0 L_\Omega^1} := \max_{t \in [0, T]} \|E(t)\|_{L^1(\Omega)}. \quad (6.171)$$

The vector space $C^0([0, T]; L^1(\Omega))$ equipped the norm $\|\cdot\|_{C_T^0 L_\Omega^1}$ is a Banach space. Let

$$\begin{aligned} \mathcal{C} := \{E : [0, T] \times \Omega \rightarrow [0, +\infty) : E \in C^0([0, T]; L^1(\Omega)), \\ 0 \leq E(t, x) \leq \max\{K(x), E^0(x)\} \quad \forall (t, x) \in (0, T) \times \Omega\}. \end{aligned} \quad (6.172)$$

The set $\mathcal{C}$ is a non-empty closed subset of $C^0([0, T]; L^1(\Omega))$. Let us define an application

$$\mathcal{Q} : e \in \mathcal{C} \mapsto E, \quad (6.173)$$

where $E : [0, T] \times \Omega \rightarrow [0, +\infty)$ is the solution of the Cauchy problem

$$\frac{\partial E}{\partial t} = \beta_E f \left(1 - \frac{E}{K}\right) \frac{\eta m}{1 + \eta m} - (\nu_E + \delta_E)E, \quad E(0) = E^0, \quad (6.174)$$

with $m \in C^0([0, T]; L^1(\Omega))$ and $f \in C^0([0, T]; L^1(\Omega))$ being the weak solutions of

$$\frac{\partial f}{\partial t} - d_1 \Delta f + \delta_F f = \nu \nu_E e \text{ in } (0, T) \times \Omega, \quad \frac{\partial f}{\partial n} = 0 \text{ on } (0, T) \times \partial\Omega, \quad f(0) = F^0, \quad (6.175)$$

$$\frac{\partial m}{\partial t} - d_2 \Delta m + \delta_M m = (1 - \nu) \nu_E e \text{ in } (0, T) \times \Omega, \quad \frac{\partial m}{\partial n} = 0 \text{ on } (0, T) \times \partial\Omega, \quad m(0) = M^0. \quad (6.176)$$

One easily checks that

$$E \in \mathcal{C}. \quad (6.177)$$

Let us point out that $(e, f, m)^T$ is a weak solution of the Cauchy problem (6.25) if and only if $\mathcal{Q}(e) = e$. We are first going to prove that $\mathcal{Q}$ is a contraction map if T is small enough, which implies that Theorem 6.3.1 holds at least if $T > 0$ is small enough. Next, we prove the existence of the solution of the Cauchy problem (6.25) for all time.

Let $\hat{e} \in \mathcal{C}$. We define $\hat{f} \in C^0([0, T]; L^1(\Omega))$ and $\hat{m} \in C^0([0, T]; L^1(\Omega))$ to be the weak solutions of

$$\frac{\partial \hat{f}}{\partial t} - d_1 \Delta \hat{f} + \delta_F \hat{f} = \nu \nu_E \hat{e} \text{ in } (0, T) \times \Omega, \quad \frac{\partial \hat{f}}{\partial n} = 0 \text{ on } (0, T) \times \partial\Omega, \quad \hat{f}(0, \cdot) = F^0(\cdot), \quad (6.178)$$

$$\frac{\partial \hat{m}}{\partial t} - d_2 \Delta \hat{m} + \delta_M \hat{m} = (1 - \nu) \nu_E \hat{e} \text{ in } (0, T) \times \Omega, \quad \frac{\partial \hat{m}}{\partial n} = 0 \text{ on } (0, T) \times \partial\Omega, \quad \hat{m}(0, \cdot) = M^0(\cdot). \quad (6.179)$$

Let $a \in C^0([0, T]; L^1(\Omega))$ be defined by

$$a := f - \hat{f}. \quad (6.180)$$

Then a is the weak solution of

$$\frac{\partial a}{\partial t} - d_1 \Delta a + \delta_F a = \nu \nu_E (e - \hat{e}) \text{ in } (0, T) \times \Omega, \quad \frac{\partial a}{\partial n} = 0 \text{ on } (0, T) \times \partial\Omega, \quad a(0, \cdot) = 0. \quad (6.181)$$

We denote by $t \in [0, +\infty) \rightarrow S_1(t) \in \mathcal{L}(L^1(\Omega); L^1(\Omega))$ the semi-group associated to $-d_1 \Delta + \delta_F \text{Id}$ with the Neumann boundary condition on $\partial\Omega$. In other words, for $\phi^0 \in L^1(\Omega)$, $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}$, $\phi(t) := S_1(t)\phi^0$ is the weak solution of

$$\frac{\partial \phi}{\partial t} - d_1 \Delta \phi + \delta_F \phi = 0 \text{ in } (0, T) \times \Omega, \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } (0, T) \times \partial\Omega, \quad \phi(0, \cdot) = \phi^0. \quad (6.182)$$

Let us recall that, for every $1 \leq p < \infty$, there exists a constant $C_0 > 0$ such that, for every $t \in (0, 1]$ and for every $\phi^0 \in L^1(\Omega)$,

$$\|S_1(t)\phi^0\|_{L^p} \leq C_0 t^{-\frac{(p-1)}{p}} \|\phi^0\|_{L^1}. \quad (6.183)$$

This property is a direct consequence of the kernel estimate given in [48, Theorem 3.2.9]; see also the proof of [38, Proposition 3.5.7] which deals with the Dirichlet boundary condition on $(0, T) \times \partial\Omega$.

From (6.181) and Duhamel's formula, we have

$$a(t) = \nu \nu_E \int_0^t S_1(t-s)(e(s) - \hat{e}(s)) ds \quad \forall t \in [0, T]. \quad (6.184)$$

Let $p \in [1, +\infty)$. From (6.183) and (6.184), one has, for all $t \in [0, T]$,

$$\begin{aligned} \|a(t)\|_{L^p} &\leq \nu \nu_E \int_0^t \|S_1(t-s)(e(s) - \hat{e}(s))\|_{L^p} ds \\ &\leq C \int_0^t (t-s)^{-\frac{(p-1)}{p}} \|e(s) - \hat{e}(s)\|_{L^1} ds \\ &\leq C \|e - \hat{e}\|_{C_T^0 L_\Omega^1} \int_0^t (t-s)^{-\frac{(p-1)}{p}} ds, \\ &\leq C \|e - \hat{e}\|_{C_T^0 L_\Omega^1} t^{\frac{1}{p}}. \end{aligned} \quad (6.185)$$

which with (6.180) leads to

$$\|f(t) - \hat{f}(t)\|_{L^p} \leq C \|e - \hat{e}\|_{C_T^0 L_\Omega^1} t^{\frac{1}{p}}. \quad (6.186)$$

In (6.185) and (6.186), and until the end of this appendix, C denotes constants which may vary from place to

place but are independent of $t \in [0, T]$, $T \in (0, 1]$, E^0 , F^0 , M^0 , e , and $\hat{e}$. Similarly,

$$\|m(t) - \hat{m}(t)\|_{L^p} \leq C \|e - \hat{e}\|_{C_T^0 L_\Omega^1} t^{\frac{1}{p}}. \quad (6.187)$$

As $e \in \mathcal{C}$, one has $e(t, x) \leq \bar{E}^0(x) := \max\{K(x), E^0(x)\}$. Therefore, using (6.175) and the maximum principle for parabolic equations,

$$0 \leq f(t, x) \leq \bar{f}(t, x) \quad \forall (t, x) \in (0, T) \times \Omega, \quad (6.188)$$

where $\bar{f} \in C^0([0, T]; L^1(\Omega))$ is the weak solution of

$$\frac{\partial \bar{f}}{\partial t} - d_1 \Delta \bar{f} + \delta_F \bar{f} = \nu \nu_E \bar{E}^0 \text{ in } (0, T) \times \Omega, \quad \frac{\partial \bar{f}}{\partial n} = 0 \text{ on } (0, T) \times \partial\Omega, \quad \bar{f}(0) = F^0. \quad (6.189)$$

From (6.189) and Duhamel's formula, one has

$$\bar{f}(t) = S_1(t)F^0 + \nu \nu_E \int_0^t S_1(t-s)\bar{E}^0 ds, \quad (6.190)$$

which, together with (6.183), implies that

$$\begin{aligned} \|\bar{f}(t)\|_{L^p} &\leq C(\|F^0\|_{L^1} t^{-\frac{(p-1)}{p}} + \|\bar{E}^0\|_{L^1} \int_0^t (t-s)^{-\frac{p-1}{p}} ds) \\ &\leq C(\|F^0\|_{L^1} t^{-\frac{p-1}{p}} + \|\bar{E}^0\|_{L^1} t^{\frac{1}{p}}), \end{aligned} \quad (6.191)$$

which, with (6.188), leads to

$$\|f(t)\|_{L^p} \leq C(\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1}) t^{-\frac{p-1}{p}}. \quad (6.192)$$

For all $t \geq 0$ and all $x \in \Omega$, we have from (6.174) (see also (6.37))

$$E(t, x) = E^0(x) e^{-\int_0^t J(s, x) ds} + \beta_E \int_0^t \frac{\eta f(s, x) m(s, x)}{1 + \eta m(s, x)} e^{-\int_s^t J(\tau, x) d\tau} ds \quad (6.193)$$

where

$$J(t, x) := (\nu_E + \delta_E) + \frac{\eta \beta_E f(t, x) m(t, x)}{K(1 + \eta m(t, x))}. \quad (6.194)$$

From this expression of E and the similar expression for $\hat{E}$ and $\hat{J}$, we have

$$\|E(t) - \hat{E}(t)\|_{L^1} \leq R_1(t) + R_2(t) + R_3(t), \quad (6.195)$$

where R_1 , R_2 , and R_3 are defined by

$$R_1(t) := \int_\Omega E^0(x) \left| e^{-\int_0^t J(s, x) ds} - e^{-\int_0^t \hat{J}(s, x) ds} \right| dx, \quad (6.196)$$

$$R_2(t) := \beta_E \int_0^t \int_\Omega \frac{\eta f(s, x) m(s, x)}{1 + \eta m(s, x)} \left| e^{-\int_s^t J(\tau, x) d\tau} - e^{-\int_s^t \hat{J}(\tau, x) d\tau} \right| dx ds, \quad (6.197)$$

$$R_3(t) := \beta_E \int_0^t \int_\Omega \left| \frac{\eta f(s, x) m(s, x)}{1 + \eta m(s, x)} - \frac{\eta \hat{f}(s, x) \hat{m}(s, x)}{1 + \eta \hat{m}(s, x)} \right| e^{-\int_s^t \hat{J}(\tau, x) d\tau} dx ds. \quad (6.198)$$

Let us now give upper bounds of R_1 , R_2 , and R_3 .

• **Upper bound of R_1 .**

For all $t \in [0, T]$ and all $x \in \Omega$, one has

$$\begin{aligned} \left| e^{-\int_0^t J(\tau, x) d\tau} - e^{-\int_0^t \hat{J}(\tau, x) d\tau} \right| &\leq \int_0^t |J(\tau, x) - \hat{J}(\tau, x)| d\tau \\ &\leq C \int_0^t f(\tau, x) |m(\tau, x) - \hat{m}(\tau, x)| d\tau \\ &\quad + C \int_0^t |f(\tau, x) - \hat{f}(\tau, x)| d\tau \end{aligned} \quad (6.199)$$

which, together with (6.196), implies that

$$R_1(t) \leq C \int_0^t \left(\int_{\Omega} E^0(x) f(\tau, x) |m(\tau, x) - \hat{m}(\tau, x)| dx \right) d\tau \\ + C \int_0^t \left(\int_{\Omega} E^0(x) |f(\tau, x) - \hat{f}(\tau, x)| dx \right) d\tau. \quad (6.200)$$

Note that

$$\frac{r-1}{2r} + \frac{r-1}{2r} + \frac{1}{r} = 1 \quad (6.201)$$

Using Hölder's inequality, (6.200) and (6.201), we have

$$R_1(t) \leq C \int_0^t \|m(\tau) - \hat{m}(\tau)\|_{L^{\frac{2r}{r-1}}} \|f(\tau)\|_{L^{\frac{2r}{r-1}}} \|E^0\|_{L^r} d\tau \\ + C \|E^0\|_{L^r} \int_0^t \|f(\tau) - \hat{f}(\tau)\|_{L^{\frac{r}{r-1}}} d\tau. \quad (6.202)$$

Using (6.187) for $p = 2r/(r-1)$, (6.192) for $p = 2r/(r-1)$, and (6.202), we get

$$R_1(t) \leq C((\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1}) \|E^0\|_{L^r} \int_0^t s^{-\frac{1}{r}} ds \|e - \hat{e}\|_{C_T^0 L_{\Omega}^1} \\ + C \|E^0\|_{L^r} \int_0^t s^{\frac{r-1}{r}} ds \|e - \hat{e}\|_{C_T^0 L_{\Omega}^1}, \quad (6.203)$$

which gives

$$R_1(t) \leq C(\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1} + 1) \|E^0\|_{L^r} T^{\frac{r-1}{r}} \|e - \hat{e}\|_{C_T^0 L_{\Omega}^1}. \quad (6.204)$$

• **Upper bound of R_2 .**

For all $0 \leq s \leq t \leq T$ and all $x \in \Omega$, we have

$$\frac{\eta f(s, x) m(s, x)}{1 + \eta m(s, x)} \leq f(s, x), \quad (6.205)$$

$$\left| e^{-\int_s^t J(\tau, x) d\tau} - e^{-\int_s^t \hat{J}(\tau, x) d\tau} \right| \leq \int_s^t |J(\tau, x) - \hat{J}(\tau, x)| d\tau = \\ \frac{\beta_E}{K} \int_s^t \left| \frac{\eta f(\tau, x) m(\tau, x)}{1 + \eta m(\tau, x)} - \frac{\eta \hat{f}(\tau, x) \hat{m}(\tau, x)}{1 + \eta \hat{m}(\tau, x)} \right| d\tau, \quad (6.206)$$

and

$$\int_s^t \left| \frac{\eta f(\tau, x) m(\tau, x)}{1 + \eta m(\tau, x)} - \frac{\eta \hat{f}(\tau, x) \hat{m}(\tau, x)}{1 + \eta \hat{m}(\tau, x)} \right| d\tau \leq \eta \int_s^t \left| f(\tau, x) \frac{m(\tau, x) - \hat{m}(\tau, x)}{(1 + \eta m(\tau, x))(1 + \eta \hat{m}(\tau, x))} \right| d\tau \\ + \int_s^t |f(\tau, x) - \hat{f}(\tau, x)| d\tau \\ \leq \eta \int_s^t f(\tau, x) |m(\tau, x) - \hat{m}(\tau, x)| d\tau \\ + \int_s^t |f(\tau, x) - \hat{f}(\tau, x)| d\tau.$$

So, using also (6.197),

$$R_2(t) \leq C \int_0^t \int_{\Omega} f(s, x) \left(\int_s^t f(\tau, x) |m(\tau, x) - \hat{m}(\tau, x)| d\tau \right) dx ds \\ + C \int_0^t \int_{\Omega} f(s, x) \left(\int_s^t |f(\tau, x) - \hat{f}(\tau, x)| d\tau \right) dx ds. \quad (6.207)$$

Then, using Hölder's inequality and Fubini's theorem, we get

$$R_2(t) \leq C \int_0^t \|f(s)\|_{L^r} \int_s^t \|f(\tau)\|_{L^{\frac{2r}{r-1}}} \|m(\tau) - \hat{m}(\tau)\|_{L^{\frac{2r}{r-1}}} d\tau ds \\ + C \int_0^t \|f(s)\|_{L^r} \int_s^t \|f(\tau) - \hat{f}(\tau)\|_{L^{\frac{r}{r-1}}} d\tau ds. \quad (6.208)$$

Using (6.192) with $p = r$ and $p = 2r/(r-1)$, (6.187) with $p = 2r/(r-1)$, and (6.186) with $p = r/(r-1)$, we get

$$R_2(t) \leq C(\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1})^2 \int_0^t s^{-\frac{r-1}{r}} \int_s^t \tau^{-\frac{1}{r}} d\tau ds \|e - \hat{e}\|_{C_T^0 L_\Omega^1} \\ + C(\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1}) \int_0^t s^{-\frac{r-1}{r}} \int_s^t \tau^{\frac{r-1}{r}} d\tau ds \|e - \hat{e}\|_{C_T^0 L_\Omega^1} \\ \leq C(\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1}) (\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1} + t) t \|e - \hat{e}\|_{C_T^0 L_\Omega^1}.$$

Hence

$$R_2(t) \leq C \left((\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1})^2 + 1 \right) T \|e - \hat{e}\|_{C_T^0 L_\Omega^1}. \quad (6.209)$$

• **Upper bound of R_3 .**

Note that, using (6.194), for all $t \in [0, T]$ and for all $x \in \Omega$ $J(t, x) \geq 0$. Hence, for all $s \leq t \in [0, T]$ and for all $x \in \Omega$,

$$e^{-\int_s^t \hat{J}(\tau, x) d\tau} \leq 1, \quad (6.210)$$

which, together with (6.198), implies that

$$R_3(t) \leq C \int_0^t \left(\int_\Omega f(s, x) |m(s, x) - \hat{m}(s, x)| dx \right) ds \\ + C \int_0^t \left(\int_\Omega |f(s, x) - \hat{f}(s, x)| dx \right) ds. \quad (6.211)$$

So

$$R_3(t) \leq C \int_0^t \|f(s)\|_{L^r} \|m(s) - \hat{m}(s)\|_{L^{\frac{r}{r-1}}} ds + C \int_0^t \|f(s) - \hat{f}(s)\|_{L^{\frac{r}{r-1}}} ds. \quad (6.212)$$

Using then (6.192) for $p = r$, (6.187) for $p = r/(r-1)$, and (6.186) for $p = r/(r-1)$, we get

$$R_3(t) \leq C(\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1}) \int_0^t ds \|e - \hat{e}\|_{C_T^0 L_\Omega^1} + C \int_0^t s^{\frac{r-1}{r}} ds \|e - \hat{e}\|_{C_T^0 L_\Omega^1} \\ \leq C(\|F^0\|_{L^1} + \|\bar{E}^0\|_{L^1} + 1) T \|e - \hat{e}\|_{C_T^0 L_\Omega^1}. \quad (6.213)$$

From (6.195), (6.204), (6.209), and (6.213), we conclude that

$$\|E - \hat{E}\|_{C_T^0 L_\Omega^1} \leq C(\|F^0\|_{L^1}^2 + \|E^0\|_{L^r}^2 + 1) T^{\frac{r-1}{r}} \|e - \hat{e}\|_{C_T^0 L_\Omega^1}, \quad (6.214)$$

which shows that $\mathcal{Q}$ is a contraction map if

$$T \leq \left(\frac{1}{C(\|F^0\|_{L^1}^2 + \|E^0\|_{L^r}^2 + 1)} \right)^{\frac{r}{r-1}}. \quad (6.215)$$

This concludes the proof of the uniqueness of the weak solution of the Cauchy problem (6.25) and the existence if $T > 0$ is small enough.

To get the existence for all time it suffices to see that if the weak solution is not defined for all time, there exists $\bar{T} > 0$ and a weak solution $(E, F, M)^T \in C^0([0, \bar{T}); L^1(\Omega)^3)$ of the Cauchy problem (6.25) such that

$$\limsup_{t \rightarrow \bar{T}^-} \|E(t)\|_{L^r} + \|F(t)\|_{L^1} = +\infty. \quad (6.216)$$

Let us check that (6.216) cannot hold. Concerning E , let us first point out that for every $T < \bar{T}$, E restricted to

$[0, T] \times \Omega$ has to be in $\mathcal{C}$. Hence,

$$0 \leq E(t, x) \leq \max\{K(x), E^0(x)\} \quad \forall (t, x) \in (0, \bar{T}) \times \Omega, \quad (6.217)$$

which implies that

$$\limsup_{t \rightarrow \bar{T}^-} \|E(t)\|_{L^r} < +\infty. \quad (6.218)$$

Next, using (6.50) and (6.217), we get

$$\|F(t)\|_{L^1} \leq \|F^0\|_{L^1} + \nu \nu_E t \int_{\Omega} \max\{K(x), E^0(x)\} dx \quad \forall t \in (0, \bar{T}), \quad (6.219)$$

which shows that

$$\limsup_{t \rightarrow \bar{T}^-} \|F(t)\|_{L^1} < +\infty. \quad (6.220)$$

Properties (6.218) and (6.220) are in contradiction with (6.216). This concludes the proof of Theorem 6.3.1. $\square$

Remark 6.8.1. *We have proven that in dimension $N = 2$, the Cauchy problem (6.25) is well-posed when $E^0 \in L^r(\Omega)$ for some $r > 1$. The same proof also works for $N = 3$ but does not work for $N \geq 4$. In dimension $N = 1$ the same proof allows to get a better result since it works also for $r = 1$. It is an open problem if in dimension $N = 2$ we also have the well-posedness for $r = 1$.*

6.9 Proof of Theorem 6.4.1 when (6.57) and (6.66) hold

In this section we assume that (6.57) and (6.66) hold and we prove that the Cauchy problem (6.58) has a unique weak solution on $[0, +\infty)$. The strategy is the same as in the proof of Theorem 6.3.1, when (6.39) holds, given in Appendix 6.8. Let us briefly explain the required modifications. For $(E, M_s) \in C^0([0, T]; L^1(\Omega) \times L^1(\Omega))$, we define

$$\|(E, M_s)\|_{C_T^0 L_{\Omega}^1 L_{\Omega}^1} := \|E\|_{C_T^0 L_{\Omega}^1} + \|M_s\|_{C_T^0 L_{\Omega}^1}. \quad (6.221)$$

The vector space $C^0([0, T]; L^1(\Omega) \times L^1(\Omega))$ equipped the norm $\|\cdot\|_{C_T^0 L_{\Omega}^1 L_{\Omega}^1}$ is a Banach space. Let

$$\begin{aligned} \tilde{\mathcal{C}} := \{ & (E, M_s) : [0, T] \times \Omega \rightarrow [0, +\infty) \times [0, +\infty) : E \in C^0([0, T]; L^1(\Omega)), M_s \in C^0([0, T]; L^1(\Omega)) \\ & 0 \leq E(t, x) \leq \max\{K(x), E^0(x)\}, \forall (t, x) \in (0, T) \times \Omega \}. \end{aligned} \quad (6.222)$$

The set $\tilde{\mathcal{C}}$ is a non-empty closed subset of $C^0([0, T]; L^1(\Omega) \times L^1(\Omega))$. Let us define an application

$$\tilde{\mathcal{Q}} : (e, m_s) \in \mathcal{C} \mapsto (E, M_s), \quad (6.223)$$

where E is the solution of the Cauchy problem

$$\frac{\partial E}{\partial t} = \beta_E f\left(1 - \frac{E}{K}\right) \frac{\eta m}{1 + \eta(m + \gamma m_s)} - (\nu_E + \delta_E)E, \quad E(0, \cdot) = E^0, \quad (6.224)$$

with $f \in C^0([0, T]; L^1(\Omega))$ and $m \in C^0([0, T]; L^1(\Omega))$ being again the weak solutions of (6.175) and (6.176) respectively, and M_s being the weak solution of

$$\begin{cases} \frac{\partial M_s}{\partial t} - d_3 \Delta M_s = u((e, f, m, m_s)^T) - \delta_s M_s & \text{in } [0, T] \times \Omega, \\ \frac{\partial M_s}{\partial n} = 0 & \text{on } [0, T] \times \partial\Omega, \\ M_s(0, \cdot) = M_s^0(\cdot). \end{cases} \quad (6.225)$$

One easily checks that

$$(E, M_s) \in \tilde{\mathcal{C}}. \quad (6.226)$$

Let us point out that $(e, f, m, m_s)^T$ is a solution of the Cauchy problem (6.58) if and only if $\tilde{\mathcal{Q}}(e, m_s) = (e, m_s)$. We are first going to prove that $\mathcal{Q}$ is a contraction map if T is small enough, which implies that Theorem 6.3.1 holds at least if $T > 0$ is small enough. Next, we prove the existence of the solution of the Cauchy problem (6.58) for all time.

Let $(\hat{e}, \hat{m}_s) \in \tilde{\mathcal{C}}$. We define $\hat{f} \in C^0([0, T]; L^1(\Omega))$ and $\hat{m} \in C^0([0, T]; L^1(\Omega))$ as in (6.178) and (6.179),

respectively, and define $\hat{M}_s \in C^0([0, T]; L^1(\Omega))$ as the weak solution of

$$\begin{cases} \frac{\partial \hat{M}_s}{\partial t} - d_3 \Delta \hat{M}_s + \delta_s \hat{M}_s = u((\hat{e}, \hat{f}, \hat{m}, \hat{m}_s)^T) \text{ in } [0, T] \times \Omega, \\ \frac{\partial \hat{M}_s}{\partial n} = 0 \text{ on } [0, T] \times \partial\Omega, \\ \hat{M}_s(0, \cdot) = M_s^0(\cdot). \end{cases} \quad (6.227)$$

Let us concentrate on the main new estimates compared to Appendix 6.8. Let us first deal with the upper bound of

$$\mathcal{M}(t) := \|M_s(t) - \hat{M}_s(t)\|_{L^1}. \quad (6.228)$$

Note that it is also for this estimate that we use the assumption $F^0 \in L^r$. We choose $\mathcal{E} \in (0, +\infty)$ such that

$$2 \max\{K(x), E^0(x)\} \leq \mathcal{E}. \quad (6.229)$$

From (6.57), (6.225), (6.227), and (6.228), we get

$$\begin{aligned} \mathcal{M}(t) \leq CT \left(\|e - \hat{e}\|_{C_T^0 L_\Omega^1} + \|f - \hat{f}\|_{C_T^0 L_\Omega^1} + \|m - \hat{m}\|_{C_T^0 L_\Omega^1} + \|m_s - \hat{m}_s\|_{C_T^0 L_\Omega^1} \right) \\ + C \int_0^t \|(f(\tau) + \hat{f}(\tau)) (|m(\tau) - \hat{m}(\tau)| + |m_s(\tau) - \hat{m}_s(\tau)|)\|_{L^1} d\tau. \end{aligned} \quad (6.230)$$

In (6.230) and until the end of this appendix, C denotes constants which may vary from place to place but are independent of $t \in [0, T]$, $T \in (0, 1]$, F^0 , M^0 , e , $\hat{e}$, m_s , and $\hat{m}_s$. Note that these constants may now depend on $\mathcal{E}$ and so on E^0 through (6.229). Let us recall that there exists a constant $C_1 > 0$ such that, for every $t \in (0, 1]$ and for every $\phi^0 \in L^r(\Omega)$,

$$\|S_1(t)\phi^0\|_{L^\infty} \leq C_1 t^{-\frac{1}{r}} \|\phi^0\|_{L^r}. \quad (6.231)$$

This property, as (6.183), is again a direct consequence of the kernel estimate given in [48, Theorem 3.2.9]; see also the proof of [38, Proposition 3.5.7] which deals with the Dirichlet boundary condition on $(0, T) \times \partial\Omega$. From (6.188), (6.190), and (6.231), we have

$$\|f(t)\|_{L^\infty} \leq C t^{-\frac{1}{r}} (\|F^0\|_{L^r} + 1). \quad (6.232)$$

Similarly,

$$\|\hat{f}(t)\|_{L^\infty} \leq C t^{-\frac{1}{r}} (\|F^0\|_{L^r} + 1). \quad (6.233)$$

From (6.186) with $p = 1$, (6.187) with $p = 1$, (6.230), (6.232), and (6.233), we get

$$\mathcal{M}(t) \leq C (\|F^0\|_{L^r} + 1) T^{\frac{r-1}{r}} \|(e - \hat{e}, m_s - \hat{m}_s)\|_{C_T^0 L_\Omega^1 L_\Omega^1}. \quad (6.234)$$

Concerning the estimate of $\|E - \hat{E}\|_{C_T^0 L_\Omega^1}$, it is almost done in the proof of (6.214). The only essential new terms that we have to bound from above are

$$H_1(t) := \int_0^t \left(\int_\Omega E^0(x) f(\tau, x) |m_s(\tau, x) - \hat{m}_s(\tau, x)| dx \right) d\tau, \quad (6.235)$$

$$H_2(t) := \int_0^t \int_\Omega f(s, x) \left(\int_s^t f(\tau, x) |m_s(\tau, x) - \hat{m}_s(\tau, x)| d\tau \right) dx ds. \quad (6.236)$$

We have already bounded from above H_1 in the proof of (6.234). (Note that $0 \leq E^0(x) \leq \mathcal{E}/2$). Let us bound from above $H_2(t)$. Using (6.232) for $r = 1$ (this inequality also holds for $r = 1$), we get

$$H_2(t) \leq C \int_0^t \|f(s)\|_{L^1} \ln \left(\frac{t}{s} \right) ds (\|F^0\|_{L^1} + 1) \|m_s - \hat{m}_s\|_{C_T^0 L_\Omega^1}, \quad (6.237)$$

which, with (6.188) and (6.191) for $p = 1$, leads to

$$H_2(t) \leq CT (\|F^0\|_{L^1} + 1)^2 \|m_s - \hat{m}_s\|_{C_T^0 L_\Omega^1}. \quad (6.238)$$

At the end, we obtain

$$\|E - \hat{E}\|_{C_T^0 L_\Omega^1} \leq C (\|F^0\|_{L^r}^2 + 1) T^{\frac{r-1}{r}} \left(\|e - \hat{e}\|_{C_T^0 L_\Omega^1} + \|m_s - \hat{m}_s\|_{C_T^0 L_\Omega^1} \right). \quad (6.239)$$

From (6.228), (6.234), and (6.239), we get

$$\|(E - \hat{E}, M_s - \hat{M}_s)\|_{C_T^0 L_\Omega^1 L_\Omega^1} \leq C (\|F^0\|_{L^r}^2 + 1) T^{\frac{r-1}{r}} \|(e - \hat{e}, m_s - \hat{m}_s)\|_{C_T^0 L_\Omega^1 L_\Omega^1}, \quad (6.240)$$

which shows that $\tilde{\mathcal{Q}}$ is a contraction map if $T > 0$ is small enough. This concludes the proof of the uniqueness of the weak solution of the Cauchy problem (6.58) and the existence of a weak solution if $T > 0$ is small enough.

To get the existence of the weak solution for all time, it suffices to see that if the weak solution is not defined for all time, there exists $\bar{T} > 0$ and a weak solution $(E, F, M, M_s)^T \in C^0([0, \bar{T}); L^1(\Omega)^4)$ of the Cauchy problem (6.58) such that

$$\limsup_{t \rightarrow \bar{T}^-} \|F(t)\|_{L^r} = +\infty. \quad (6.241)$$

However, using the second, fifth, and sixth lines of (4.5) and Duhamel's formula, we get

$$\bar{f}(t) = S_1(t)F^0 + \nu\nu_E \int_0^t S_1(t-s)\bar{E}^0 ds \quad \forall t \in [0, \bar{T}), \quad (6.242)$$

which gives the existence $C(\bar{T}) \in (0, +\infty)$ such

$$\|F(t)\|_{L^r} \leq C(\bar{T})(\|F^0\|_{L^r} + 1) \quad \forall t \in [0, \bar{T}), \quad (6.243)$$

in contradiction with (6.241). This concludes the proof of Theorem 6.3.1. $\square$

6.10 Proof of the existence of a weak solution

In this section, we prove the existence of a weak solution for initial data in L^1 . Here, we focus solely on the 4-dimensional closed-loop Cauchy problem (6.58), which is the most challenging case. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of elements of $C^0([0, +\infty)^4; [0, +\infty))$ such that, for every $n \in \mathbb{N}$

$$0 \leq u_n((E, F, M, M_s)^T) \leq C_u(1 + E + F + M + M_s) \quad \forall (E, F, M, M_s)^T \in [0, +\infty)^4, \quad (6.244)$$

$$u_n(y) = 0, \quad \forall y = (E, F, M, M_s)^T \in [0, +\infty)^4 \text{ such that } E + F + M + M_s \geq n, \quad (6.245)$$

and such that

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_{C^0(\mathcal{K})} = 0 \text{ for every compact set } \mathcal{K} \subset [0, +\infty)^4. \quad (6.246)$$

Clearly such a sequence $(u_n)_{n \in \mathbb{N}}$ exists. Let $(E^0, F^0, M^0, M_s^0)^T : \Omega \rightarrow [0, +\infty)^4$ be such that (6.65) holds. Then, let $((E_n^0, F_n^0, M_n^0, M_{sn}^0)^T)_{n \in \mathbb{N}}$ be a sequence of maps from Ω into $[0, +\infty)^4$ such that, for every $n \in \mathbb{N}$,

$$E_n^0 \in L^\infty(\Omega), \quad F_n^0 \in L^{r_n}(\Omega) \text{ for some } r_n > 1, \quad M_n^0 \in L^1(\Omega), \text{ and } M_{sn}^0 \in L^1(\Omega), \quad (6.247)$$

and such that, as $n \rightarrow +\infty$,

$$E_n^0 \rightarrow E^0 \quad \text{in } L^1(\Omega), \quad (6.248)$$

$$F_n^0 \rightarrow F^0 \quad \text{in } L^1(\Omega), \quad (6.249)$$

$$M_n^0 \rightarrow M^0 \quad \text{in } L^1(\Omega), \quad (6.250)$$

$$M_{sn}^0 \rightarrow M_s^0 \quad \text{in } L^1(\Omega). \quad (6.251)$$

Again, such a sequence $((E_n^0, F_n^0, M_n^0, M_{sn}^0)^T)_{n \in \mathbb{N}}$ exists. Let $T > 0$. For $n \in \mathbb{N}$, let $y_n = (E_n, F_n, M_n, M_{sn})^T$ be the weak solution on $[0, T]$ of the following closed-loop system:

$$\frac{\partial E_n}{\partial t} = \beta_E F_n \left(1 - \frac{E_n}{K(x)}\right) - (\nu_E + \delta_E) E_n, \quad t \geq 0, x \in \Omega, \quad (6.252)$$

$$\frac{\partial F_n}{\partial t} - d_1 \Delta F_n + \delta_F F_n = \nu \nu_E E_n, \quad t \geq 0, x \in \Omega, \quad (6.253)$$

$$\frac{\partial M_n}{\partial t} - d_2 \Delta M_n + \delta_M M_n = (1 - \nu) \nu_E E_n, \quad t \geq 0, x \in \Omega, \quad (6.254)$$

$$\frac{\partial M_{sn}}{\partial t} - d_3 \Delta M_{sn} + \delta_s M_{sn} = u(E_n, F_n, M_n, M_{sn}), \quad t \geq 0, x \in \Omega, \quad (6.255)$$

$$\frac{\partial F_n}{\partial n} = \frac{\partial M_n}{\partial n} = \frac{\partial M_{sn}}{\partial n} = 0, \quad t \geq 0, x \in \partial\Omega, \quad (6.256)$$

satisfying the initial condition

$$y_n(0) = (E_n^0, F_n^0, M_n^0, M_{sn}^0)^T. \quad (6.257)$$

The existence (and uniqueness) of y_n is proved in Appendix 6.9.

Note that, almost everywhere in Q_T ,

$$E_n(t, x) \leq \max(K(x), E_n^0(x)) \leq K(x) + E_n^0(x). \quad (6.258)$$

Since $K \in L^1(\Omega)$, using (6.248) and (6.258), one has

$$\|E_n(t)\|_{L^1(\Omega)} \leq C \quad \forall t \in [0, T]. \quad (6.259)$$

In (6.259) and throughout all Appendix 6.10, C denotes various positive constants which may vary from place to place but are independent $n \in \mathbb{N}$ and $t \in [0, T]$. However, even if it is not the case for (6.259), they may depend on T , which is fixed in this proof. From (6.50) with $F = F^n$ and $E = E^n$, (6.249), (6.257), and (6.259)

$$\|F_n(t)\|_{L^1(\Omega)} \leq C \quad \forall t \in [0, T]. \quad (6.260)$$

Similarly, one has

$$\|M_n(t)\|_{L^1(\Omega)} \leq C \quad \forall t \in [0, T]. \quad (6.261)$$

Using (6.244), (6.259), (6.260), and (6.261),

$$\int_{\Omega} u_n((E_n, F_n, M_n, M_{sn}^T)(t)) \leq C \left(1 + \int_{\Omega} M_{sn}(t)\right). \quad (6.262)$$

Using (6.64) for $\varphi = 1$, $M_s = M_{sn}$, $u((E, F, M, M_s)^T) = u_n((E_n, F_n, M_n, M_{sn}^T)$, (6.262), and $M_s^0 = M_{sn}^0$, together with (6.251), (6.262) and the Gronwall lemma in the integral form,

$$\|M_{sn}(t)\|_{L^1(\Omega)} \leq C \quad \forall t \in [0, T]. \quad (6.263)$$

Using (6.249), (6.253), (6.256), (6.259) and a classical compactness property (see, for example, [87, Lemma 5.6]), we deduce the existence of $F \in L^1((0, T); W^{1,1}(\Omega))$ such that, up to the extraction of a subsequence,

$$F_n \rightarrow F \text{ in } L^1((0, T); W^{1,1}(\Omega)) \text{ as } n \rightarrow +\infty. \quad (6.264)$$

Similarly, one has the existence of $M \in L^1((0, T); W^{1,1}(\Omega))$ and $M_s \in L^1((0, T); W^{1,1}(\Omega))$ such that, up to the extraction of subsequences,

$$M_n \rightarrow M \text{ in } L^1((0, T); W^{1,1}(\Omega)) \text{ as } n \rightarrow +\infty, \quad (6.265)$$

$$M_{sn} \rightarrow M_s \text{ in } L^1((0, T); W^{1,1}(\Omega)) \text{ as } n \rightarrow +\infty. \quad (6.266)$$

Note that (6.264), (6.265), and (6.266) imply, up to the extraction of a subsequence,

$$F_n(t, x) \rightarrow F(t, x) \text{ as } n \rightarrow +\infty, \text{ for almost every } (t, x) \in (0, T) \times \Omega, \quad (6.267)$$

$$M_n(t, x) \rightarrow M(t, x) \text{ as } n \rightarrow +\infty, \text{ for almost every } (t, x) \in (0, T) \times \Omega, \quad (6.268)$$

$$M_{sn}(t, x) \rightarrow M_s(t, x) \text{ as } n \rightarrow +\infty, \text{ for almost every } (t, x) \in (0, T) \times \Omega, \quad (6.269)$$

and the existence of $\Phi \in L^1((0, T) \times \Omega)$ such that, for every $n \in \mathbb{N}$,

$$F_n(t, x) \leq \Phi(t, x) \text{ for almost every } (t, x) \in (0, T) \times \Omega, \quad (6.270)$$

$$M_n(t, x) \leq \Phi(t, x) \text{ for almost every } (t, x) \in (0, T) \times \Omega, \quad (6.271)$$

$$M_{sn}(t, x) \leq \Phi(t, x) \text{ for almost every } (t, x) \in (0, T) \times \Omega. \quad (6.272)$$

In particular, there exists a set $\mathcal{N}$ of Lebesgue measure 0 such that, for every $t \in [0, T] \setminus \mathcal{N}$,

$$F_n(t, x) \rightarrow F(t, x) \text{ as } n \rightarrow +\infty, \text{ for almost every } x \in \Omega, \quad (6.273)$$

$$M_n(t, x) \rightarrow M(t, x) \text{ as } n \rightarrow +\infty, \text{ for almost every } x \in \Omega, \quad (6.274)$$

$$M_{sn}(t, x) \rightarrow M_s(t, x), \quad (6.275)$$

$$F_n(t, x) \leq \Phi(t, x) \text{ for almost every } x \in \Omega, \quad (6.276)$$

$$M_n(t, x) \leq \Phi(t, x) \text{ for almost every } x \in \Omega, \quad (6.277)$$

$$M_{sn}(t, x) \leq \Phi(t, x) \text{ for almost every } x \in \Omega. \quad (6.278)$$

Let

$$A_n(t, x) := \beta_E F_n(t, x) \frac{\eta M_n(t, x)}{1 + \eta(M_n(t, x) + \gamma M_{sn}(t, x))}, \quad (6.279)$$

$$A(t, x) := \beta_E F(t, x) \frac{\eta M(t, x)}{1 + \eta(M(t, x) + \gamma M_s(t, x))}, \quad (6.280)$$

$$B_n(t, x) := \frac{\beta_E}{K(x)} F_n(t, x) \frac{\eta M_n(t, x)}{1 + \eta(M_n(t, x) + \gamma M_{sn}(t, x))} + (\nu_E + \delta_E), \quad (6.281)$$

$$B(t, x) := \frac{\beta_E}{K(x)} F(t, x) \frac{\eta M(t, x)}{1 + \eta(M(t, x) + \gamma M_s(t, x))} + (\nu_E + \delta_E). \quad (6.282)$$

From our definition of y_n , one has

$$E_n(t, x) = e^{-\int_0^t B_n(s, x) ds} E_n^0(x) + \int_0^t e^{-\int_s^t B_n(\tau, x) d\tau} A_n(s, x) ds. \quad (6.283)$$

Let

$$E(t, x) = e^{-\int_0^t B(s, x) ds} E^0(x) + \int_0^t e^{-\int_s^t B(\tau, x) d\tau} A(s, x) ds. \quad (6.284)$$

Note that

$$E \in C^0([0, T]; L^1(\Omega)). \quad (6.285)$$

From (6.248), extracting a subsequence if necessary,

$$E_n^0(x) \rightarrow E^0(x) \text{ for almost every } x \in \Omega \quad (6.286)$$

and there exists $\Psi : \Omega \rightarrow [0, +\infty)$ in $L^1(\Omega)$ such that

$$E_n^0(x) \leq \Psi(x) \text{ for almost every } x \in \Omega. \quad (6.287)$$

From (6.267) to (6.272), (6.279) to (6.284), (6.286), and (6.287),

$$E_n(t, x) \rightarrow E(t, x) \text{ for almost every } (t, x) \in (0, T) \times \Omega, \text{ as } n \rightarrow +\infty \quad (6.288)$$

and there exists $\Psi_1 : \Omega \rightarrow [0, +\infty)$ in $L^1(\Omega)$ such that, for every $n \in \mathbb{N}$,

$$E_n(t, x) \leq \Psi_1(x) \text{ for almost every } x \in \Omega. \quad (6.289)$$

Let $\varphi : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ be of class $\mathcal{C}^1$. Let $t \in [0, T]$. From our definition of y_n , one has

$$\begin{aligned} \int_{\Omega} F_n(t, x) \varphi(t, x) dx - \int_{\Omega} F_n^0(x) \varphi(0, x) dx + d_1 \int_{Q_t} \nabla F_n \cdot \nabla \varphi - \int_{Q_t} F_n \frac{\partial \varphi}{\partial t} = \\ \int_{Q_t} (\nu \nu_E E_n - \delta_F F_n) \varphi. \end{aligned} \quad (6.290)$$

Letting $n \rightarrow +\infty$ in (6.290) and using (6.249), (6.264), (6.273), (6.276), (6.288), and (6.287), one gets

$$(6.26) \text{ holds provided that } t \in [0, T] \setminus \mathcal{N}. \quad (6.291)$$

Assume for the moment that the following lemma holds.

Lemma 6.10.1. *Let $\mathcal{N} \subset [0, T]$ be of Lebesgue measure 0. Let $f \in L^1((0, T); W^{1,1}(\Omega))$ be such that, for every $\varphi : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ of class $\mathcal{C}^1$,*

$$\int_{\Omega} f(t, x) \varphi(t, x) dx + d_1 \int_{Q_t} \nabla F \cdot \nabla \varphi - \int_{Q_t} f \frac{\partial \varphi}{\partial t} + \delta_F \int_{Q_t} F \varphi = 0 \quad \forall t \in [0, T] \setminus \mathcal{N}. \quad (6.292)$$

Then $f = 0$.

Let

$$\bar{F}(t) = S_1(t)F_0 + \int_0^t S_1(t-s)E(s)ds, \quad (6.293)$$

where S_1 is defined in Appendix 6.8; see, in particular, (6.182). Note $f := F - \bar{F}$ satisfies (6.292). As a consequence of Lemma 6.10.1, $F = \bar{F} \in C^0[0, T]; L^1(\Omega)$ and, by density and the continuity with respect to t of all the terms of (6.26), one has (6.26) for every $t \in [0, T]$. Similar proofs show that M and M_s are in $C^0([0, T]; L^1(\Omega))$ and that (6.27) and (6.64) holds for every $t \in [0, T]$. Hence $(E, F, M, M_s)^T$ is a weak solution of the closed-loop Cauchy problem (6.58) for the initial condition $(E(0), F(0), M(0), M_s(0))^T = (E^0, F^0, M^0, M_s^0)^T$.

It remains to prove Lemma 6.10.1. Let $g \in L^1((0, T); W^{1,1}(\Omega))$ be defined by

$$g(t, x) := \int_0^t f(s, x) ds. \quad (6.294)$$

One easily sees that $g \in C^0([0, T]; L^1(\Omega))$ and is such that, for every $\varphi : [0, T] \times \bar{\Omega} \rightarrow \mathbb{R}$ of class $\mathcal{C}^1$,

$$\int_{\Omega} g(t, x) \varphi(t, x) dx + d_1 \int_{Q_t} \nabla g \cdot \nabla \varphi - \int_{Q_t} g \frac{\partial \varphi}{\partial t} + \delta_F \int_{Q_t} g \varphi = 0 \quad \forall t \in [0, T], \quad (6.295)$$

which implies that $g = 0$ and therefore $f = 0$. This ends the proof of Lemma 6.10.1.

Hence, for every initial data $y^0 : \Omega \rightarrow [0, +\infty)^4$ in $L^1(\Omega)^4$ and for every $T > 0$, there exists a weak solution on $[0, T]$ of the closed-loop Cauchy problem (6.58). This existence can be extended to $[0, +\infty)$. Indeed, let $y^0 : \Omega \rightarrow [0, +\infty)^4$ be in $L^1(\Omega)^4$ and let $T_1 = 1$. There exists a weak solution $y_1 \in C^0([0, T_1]; L^1(\Omega))^4$ of the closed-loop Cauchy problem (6.58) on $[0, T_1]$ for the initial condition y^0 . Let $T_2 = T_1 + 1$. We now start at T_1 with the initial condition $y_1(T_1)$. There exists a weak solution y_2 on $[T_1, T_2]$ of the closed-loop Cauchy problem (6.58) for this initial condition. Let $y \in C^0([0, T_2]; L^1(\Omega))$ be defined by

$$y(t) = y_1(t) \text{ for } t \in [0, T_1] \text{ and } y(t) = y_2(t) \text{ for } t \in [T_1, T_2]. \quad (6.296)$$

Then, as one easily checks, y is a weak solution of the closed-loop Cauchy problem (6.58) on $[0, T_2]$ for the initial condition y^0 . We keep going: consider $T_n = n$ and finally we get the existence of a weak solution of the closed-loop Cauchy problem (6.58) on $[0, +\infty)$ for the initial condition y^0 .

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Abstract

The sterile insect technique (SIT) has recently become one of the methods used for controlling mosquito populations. This technique involves the release of sterile mosquitoes that compete with wild males for mating with females. The mathematical study of this control system aims to develop strategies that optimize the release of sterile insects, stabilize the population to zero, and prevent resurgence. This is the objective of the present work. The challenge of stabilizing the system to zero lies in the fact that, as the population approaches zero, fewer releases are needed to maintain the population close to zero asymptotically. However, sufficient sterile insects must still be released to counteract, in particular, disturbances in the dynamic parameters. In control theory, many tools exist to address this problem, but they often fail to ensure the positivity of the control function.

In this work, we apply the backstepping method to design a nonnegative feedback control law that addresses the problem of robustness and global asymptotic stability of the dynamics to zero. Additionally, we study other linear feedback control laws that can globally stabilize the control system. However, implementing these feedback control laws requires measuring the system states. We address this issue by designing an observer that, based on measurements of sterile and wild male mosquitoes, enables the estimation of the other components of the state of the system.

The implementation of SIT for controlling mosquito populations in practical situations also requires considering the spatial dispersion of adult mosquitoes. Recent advance in using drones to release sterile male mosquitoes have improved the precision of the release, which helps improve their dispersion.

In our study, we consider a system of reaction-diffusion equations that models the spatial dispersion of mosquito populations in a two-dimensional bounded and smooth domain. We study sterile male releases within this framework. Under Neumann boundary conditions and assuming the same diffusion coefficient for sterile and wild males, we design nonnegative feedback laws which globally asymptotically stabilize the population to zero over the entire domain.

Keywords: Vector borne disease, Mosquito population control, Dynamical control system, Feedback design, Backstepping feedback, Observer design, Lyapunov stability, Reaction-diffusion equations

Résumé

La technique des insectes stériles (TIS) est récemment devenue l'une des méthodes utilisées pour le contrôle des populations de moustiques. Cette technique consiste à relâcher des moustiques mâles stériles qui entrent en compétition avec les mâles sauvages pour s'accoupler avec les femelles.

L'étude mathématique de ce système de contrôle vise à développer des stratégies optimisant le lâcher d'insectes stériles, stabilisant la population près de zéro et empêchant toute résurgence. Le défi de stabiliser le système autour de zéro réside dans le fait que, à mesure que la population approche de zéro, moins de lâchers sont nécessaires pour la maintenir asymptotiquement proche de zéro. Cependant, il est crucial de relâcher un nombre suffisant d'insectes stériles pour contrer, en particulier, les perturbations des paramètres dynamiques. En théorie du contrôle, de nombreux outils existent pour résoudre ce problème, mais ils ne garantissent pas souvent la positivité de la fonction de contrôle.

Dans ce travail, nous appliquons la méthode de backstepping pour concevoir une loi de contrôle feedback positive, répondant au problème de robustesse et de stabilité asymptotique globale des dynamiques vers zéro. De plus, nous étudions d'autres lois de contrôle feedback linéaires capables de stabiliser globalement et asymptotiquement le système de contrôle vers zéro. Cependant, la mise en œuvre de ces lois de contrôle nécessite de mesurer les états du système. Nous abordons ce problème en concevant un observateur qui, à partir des mesures des moustiques mâles stériles et des mâles sauvages, permet d'estimer les autres composantes de l'état du système.

La mise en œuvre de la TIS pour contrôler les populations de moustiques dans des situations pratiques nécessite également de prendre en compte la dispersion spatiale des moustiques adultes. Les avancées récentes dans l'utilisation de drones pour relâcher les mâles stériles ont amélioré leur dispersion.

Dans notre étude, nous considérons des systèmes d'équations de réaction-diffusion modélisant la dispersion spatiale des populations de moustiques dans un domaine borné et lisse à deux dimensions. Nous étudions les lâchers de moustiques stériles dans ce cadre. Sous des conditions aux limites de Neumann et en supposant le même coefficient de diffusion pour les mâles stériles et sauvages, nous avons construit des lois de contrôle feedback stabilisant globalement et asymptotiquement la population vers zéro sur l'ensemble du domaine.

Mots clés : Maladie vectorielle, Contrôle de la population de moustiques, système dynamique de contrôle, Conception de feedback, Methode backstepping, conception d'observateur, Stabilité au sens de Lyapunov, Equations de réaction-diffusion
